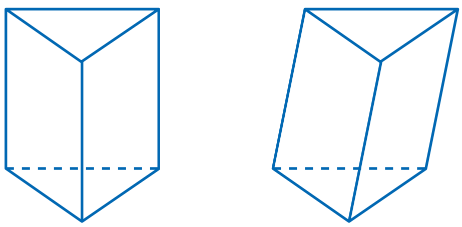
LESSON 17

Three-Dimensional Figures

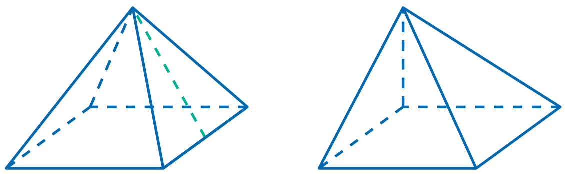
Polyhedra and Other Solids

UNDERSTAND A two-dimensional figure, such as a square, has two dimensions: length and width. A three-dimensional (or solid) object, such as a cube, has three dimensions: length, width, and height. The length and width of a solid generally refer to the dimensions of its base. A **polyhedron** (plural: polyhedra) is a three-dimensional figure bounded by polygons, called faces. The line segments where the faces intersect are called edges, and the points where three or more edges meet are called vertices.

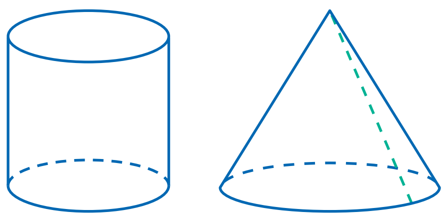
A **prism** is a polyhedron having two congruent, parallel bases and parallelograms for all other faces. The faces that are not bases, the parallelograms, are called lateral faces. If one base of a prism lies directly above the other base, then the lateral faces are rectangles and the figure is a right prism. If not, the figure is an oblique prism. A prism is categorized by the shape of its bases. For example, a prism with triangles for bases is a triangular prism.



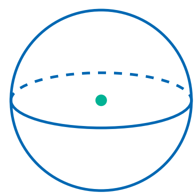
A **pyramid** is a polyhedron with one base, which can be any polygon. Like prisms, pyramids are also categorized by their bases. For example, a pyramid with a square base is a square pyramid. The lateral faces are triangles that meet at a common vertex, called the apex. If the apex of a pyramid is directly above the center of its base, then it is a right pyramid. Otherwise, it is an oblique pyramid. The slant height of a pyramid is the distance from the base to the apex along the center of a lateral face. In other words, it is the altitude of a triangular face.



If a solid figure has any curved edge or face, it is not a polyhedron. A **cylinder** has two congruent bases, like a prism, but the bases are circles. A **cone** is similar to a pyramid in that it has a single base and an apex, but its base is a circle and its lateral surface is curved. Cones, like pyramids, have slant height. Like prisms and pyramids, cylinders and cones can be right or oblique.

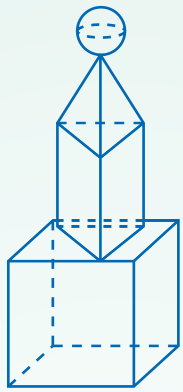


Recall that a circle is the set of all points in a plane that are a given distance from the center. A **sphere** is the set of all points in three-dimensional space that are a given distance from a point, also called the center. Like a circle, a sphere has radii, chords, and diameters. A sphere is a continuous curved surface, so it has no faces, edges, or vertices.



Connect

The monument shown below is made up of several different solid figures. Identify each of them.



1 Identify the base.

The base is a rectangular prism in which each face is a square. This figure is called a cube.

2 Identify the solid resting on the base.

The solid resting on the base has two congruent triangular bases and lateral faces that are rectangles. This solid is a triangular prism.

3 Identify the next solid up.

The next solid also has a triangular base, but it has only one. Its lateral faces are triangles that meet at a single point. This solid is a triangular pyramid.

4 Identify the solid on top.

The solid on top has no faces, edges, or vertices. It is made up of a continuous, curved surface. The solid on top is a sphere.

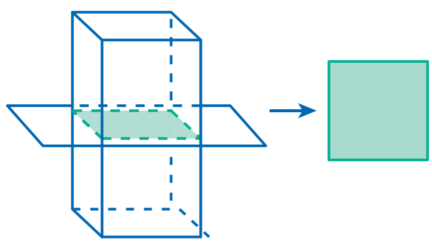
DISCUSS

Which of the solids in the monument are right? oblique?

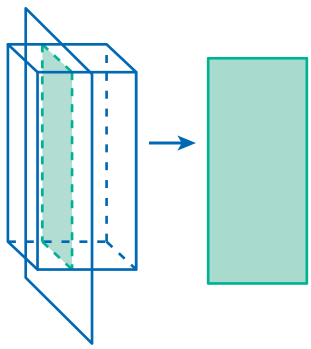
Cross Sections and Rotations

UNDERSTAND Imagine slicing a plane through a three-dimensional solid. The portion of solid inside that plane would be a two-dimensional shape called a **cross section**.

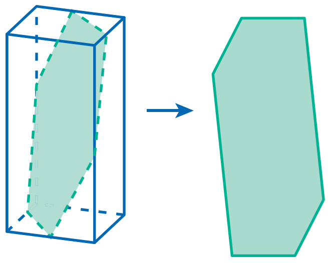
Consider the square prism below. A plane parallel to the bases produces a square cross section, identical to the bases. A plane perpendicular to the bases produces a rectangular cross section. Other planes produce other shapes, such as the hexagon shown.



Plane parallel to the base



Plane perpendicular to the base

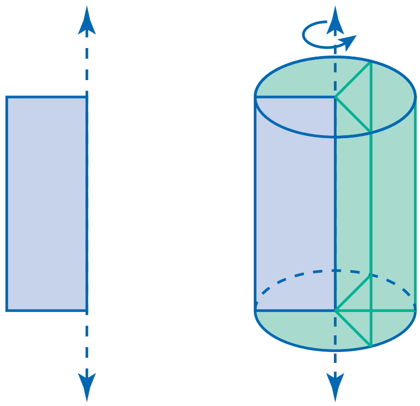


Plane neither parallel nor perpendicular to the base

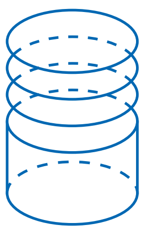
Every cross section of a polyhedron is a polygon. Figures that are not polygons, such as cylinders and cones, have some cross sections that are circles or ellipses. For prisms and cylinders, all cross sections made parallel to the bases are congruent to the bases. For pyramids and cones, all cross sections made parallel to the base are similar to the base but different in size.

UNDERSTAND You can produce a three-dimensional solid by rotating a two-dimensional figure around a line called an axis. This is not the same as the rotational transformations that were done on the plane. During those transformations, the points of the figure moved in the plane. During this type of rotation, each point is spread over an area of space.

The rectangle on the right is rotated 360° around one of its sides. The result is a cylinder. Notice how the left vertices become the circular edges of the bases.



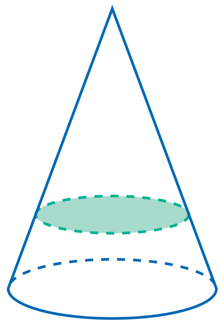
UNDERSTAND You can think of a solid as a stack of cross sections. For example, in a cylinder, all cross sections made parallel to the bases will be congruent circles. You can imagine "building" a cylinder by stacking these flat circles, like a roll of very thin coins. For a prism or cone, the stacked shapes would be similar, but not congruent. The closer a cross section is to the apex, the smaller it would be.



Connect

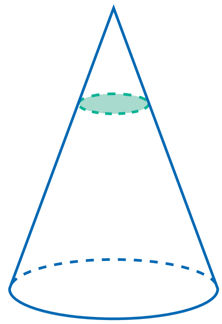
Take several cross sections of a right cone. Identify the shape of each cross section and compare it to the base of the cone.

1 Take a cross section parallel to the base.



Any cross section parallel to the base of a cone is a circle. Since the base is a larger circle, the cross section is similar to the base.

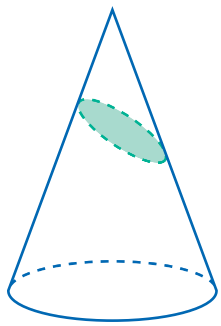
2 Take another cross section parallel to the base, but at a different height.



This cross section is also a circle, but because it was taken closer to the apex, it is smaller than the previous cross section. All circles are similar, so the cross section is similar to the base as well as the previous cross section.

3 Take a cross section at an angle to the base.

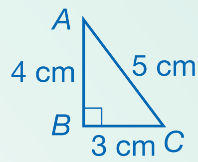
The cross section taken at this angle produces an ellipse. It is not similar to the base.



TRY

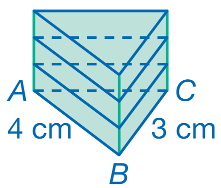
Take a cross section perpendicular to the base and through the apex. Describe the shape of the cross section.

EXAMPLE A Use $\triangle ABC$ to create solid figures in the following ways. Identify the solids created.



- Stack congruent triangles on top of each other.
- Stack increasingly smaller similar triangles on top of each other.
- Rotate the triangle 360° around $\overline{AB}$.

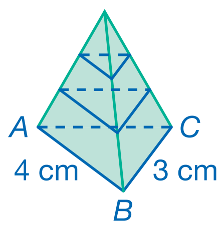
1 Stack congruent triangles to create a solid.



All triangles in the stack are congruent, so every horizontal cross section will be congruent to $\triangle ABC$. This solid is a triangular prism whose base is $\triangle ABC$.

2 Stack similar triangles to create a solid.

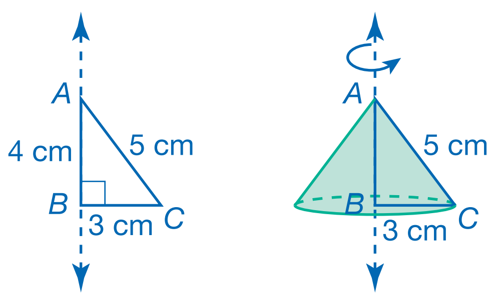
Each similar triangle becomes smaller as we go up from the base. Eventually, it will become so small that it will be a single point.



A solid with a triangular base and flat sides that meet at a point is a triangular pyramid. The base of the pyramid is $\triangle ABC$.

3 Rotate the triangle to create a solid.

Rotate 360° around $\overline{AB}$.



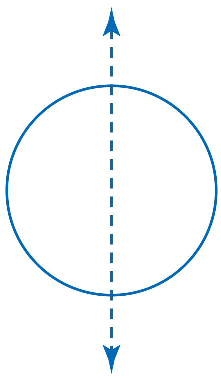
The solid has a circular base and a curved lateral face that tapers to a point, so it is a cone. The base has a radius equal to the length of $\overline{BC}$. The height is equal to the length of $\overline{AB}$. The slant height is equal to the hypotenuse, 5 cm.

DISCUSS

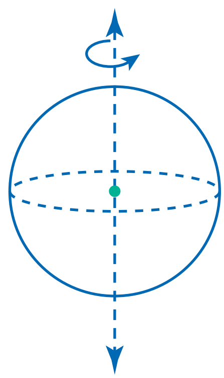
Could you create an oblique prism or pyramid by stacking triangles? Could you create an oblique cone by rotating a triangle around one of its sides?

EXAMPLE B Rotate a circle 360° about its diameter and identify the solid created. Then, take cross sections of the solid and identify their shapes.

1 Sketch a circle and extend one of its diameters.

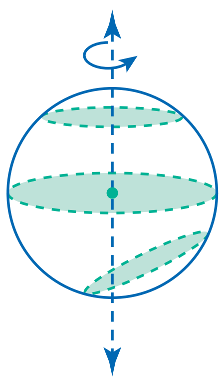


2 Rotate the circle about the diameter.



▶ The resulting solid is a sphere with the same diameter and radius as the circle.

3 Take several cross sections.

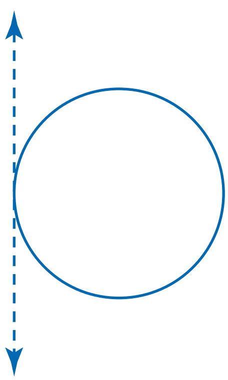


Every cross section is a circle, though not all are the same size. The largest possible circular cross sections, which contain the center, are congruent to the original circle that was rotated to create the sphere.

▶ All of the cross sections of a sphere are circles.

DISCUSS

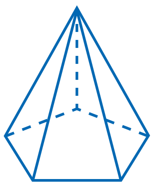
Imagine rotating a circle around a line through only one of its points. What shape results from this rotation?



Practice

Identify the cross section that results from slicing through each solid as indicated.

1. Plane parallel to the base



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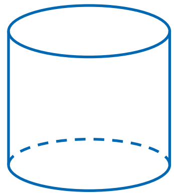
2. Plane perpendicular to the base



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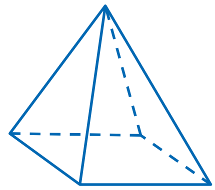
3. Plane perpendicular to the base through the apex



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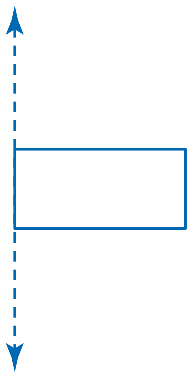
HINT A cross section made perpendicular to the base will resemble a front view of the solid.

Identify the solid that will be formed by rotating the figure 360° around the dashed line. Sketch the result and name or describe the solid.

4.



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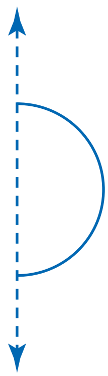

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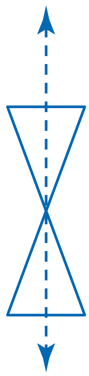

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HINT A rotated figure may be a combination of two or more solids that you studied.

7. What solid would be created by stacking many congruent hexagons, one on top of another?

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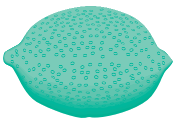
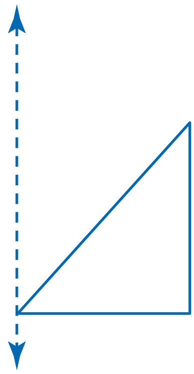
8. A café owner slices limes to put in water and iced tea. She slices some lengthwise and others widthwise. Sketch two types of slices that could result.



9. **DESCRIBE** What solid will result from rotating the triangle below 360° around the dashed line?



Describe the solid. ______________________________

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Describe a cross section taken parallel to the base of the solid. ______________________________

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Sketch an example of this cross section.

10. **THINK CRITICALLY** Is it possible to produce a polyhedron by rotating a figure around a line? Why or why not?

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