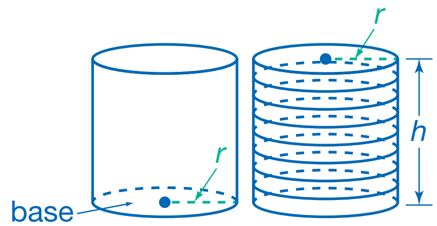
LESSON 18

Volume Formulas

Volume and Cylinders

UNDERSTAND The **volume** of a solid object is the amount of space that it occupies. It is measured in cubic units.

Recall that every cross section of a cylinder is a circle congruent to the bases, and that you can think of a cylinder as a stack of circles. If the height of the cylinder is h units, you would stack the circles h units high. So, to find the volume of a cylinder, multiply the area of the circular base by the height.

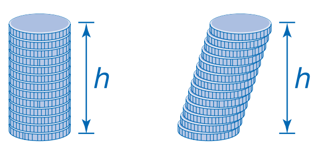


$$V = \pi r^2 h$$

This formula is the same for both a right cylinder and an oblique cylinder. However, remember that the height must be measured perpendicular to the bases.

Right Cylinder Oblique Cylinder

To understand why the formula is the same for both cylinders, think of a stack of coins. Each coin has a certain volume, and the volume of the stack is the sum of the volumes of the coins. Whether the coins are stacked straight up or displaced to form an oblique cylinder, the volume of each coin stays the same, so the volume of the stack remains the same.

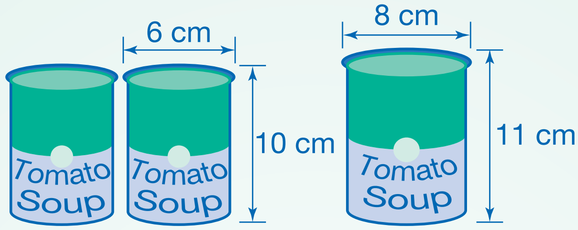


Imagine cutting a cross section from each cylinder at the same height and parallel to the bases. If the cross sections have the same area and the cylinders have the same height, the cylinders must have the same volume.

The volume of a cylinder equals the area of the base multiplied by its height, $V = Bh$. This formula can also be used to find the volume of any prism.

Connect

Dayshawn will buy either the two small cans of tomato soup shown or the one family-size can. Which would provide more soup? Approximately how much more?



1 Find the total volume of the two small cans.

The diameter of each small can is 6 cm. The radius is half the length of the diameter, so $r = 3$ cm. The cans have a height, h, of 10 cm.

$V = Bh = \pi r^2 h$

$V = \pi(3)^2(10)$

$V = 90\pi$

$V \approx 282.60 \text{ cm}^3$

The volume of two cans would be about $2 \cdot 282.60$, or 565.20 cm^3.

2 Find the volume of the family-size can.

The diameter of the family-size can is 8 cm, so $r = 4$ cm. Its height, h, is 11 cm.

$V = \pi r^2 h$

$V = \pi(4)^2(11)$

$V = 176\pi$

$V \approx 552.64 \text{ cm}^3$

3 Compare the volumes.

$565.20 > 552.64$

$565.20 - 552.64 = 12.56$

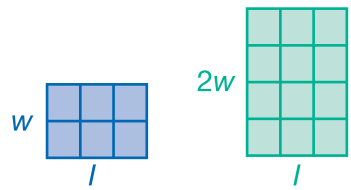
▶ Buying two small cans would provide approximately 12.56 cubic centimeters more soup.

TRY

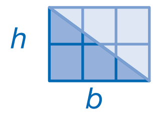
The volume of a large cylindrical can of vegetable broth is 200 cm^3. If the radius of the can is tripled, how much broth could the new can hold?

Dimensions and Formulas

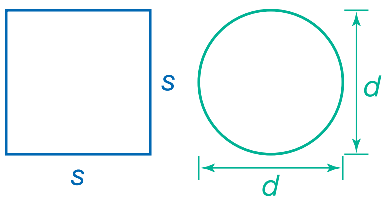
UNDERSTAND The area of a two-dimensional figure is proportional to its two dimensions: length and width. For example, the area of a rectangle is equal to the product of its length and its width. If its length or width is doubled, its area will also be doubled. So, the area is proportional to both the length and width. This means that there is a constant of proportionality, k_{rect}, such that $A_{\text{rect}} = k_{\text{rect}} \cdot l \cdot w$. In this case, $k_{\text{rect}} = 1$.



Draw a diagonal within one of the rectangles. This divides the rectangle into two congruent triangles. Since the area of the rectangle is divided between the two triangles, the area of each triangle must be equal to half of the area of the rectangle. If either the base length or height of a triangle is doubled, the area will also be doubled, just as it is with a rectangle. So, as with a rectangle, the area of a triangle is proportional to both its base length and its height: $A_{\text{tri}} = k_{\text{tri}} \cdot b \cdot h$. This area is half that of the rectangle, so $k_{\text{tri}} = \frac{1}{2}$, and $A_{\text{tri}} = \frac{1}{2} \cdot b \cdot h$.



Since all sides of a square are congruent, the same length is used to measure both dimensions. So, $A_{\text{squ}} = k_{\text{squ}} \cdot s \cdot s = k_{\text{squ}} \cdot s^2$. The constant of proportionality, k_{squ}, is 1, just as for the rectangle. According to this formula, $A_{\text{squ}} = s^2$, if the side length is doubled (or increased by a factor of 2), the area will increase by a factor of 2^2, or 4.



In a circle, the diameter is the same across the figure's width and its length. As with the side of a square, if the diameter is doubled, the area will be quadrupled. A circle is more commonly defined by its radius than its diameter, though. Doubling the radius of a circle will also quadruple its area, just as with the diameter. Therefore, the circle must have an area formula similar to that of the square: $A_{\text{circ}} = k_{\text{circ}} \cdot r^2$. In a previous lesson, you derived the formula for area of a circle: $A_{\text{circ}} = \pi r^2$. Thus, the constant of proportionality, k_{circ}, relating the area of a circle to its radius squared is π.

The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Notice that the radius is raised to the second power in the circle's area formula and to the third power in the sphere's volume formula.

As you saw above, the area of a plane figure is proportional to its two dimensions. The volume of a three-dimensional figure is proportional to its three dimensions: length, width, and height. For a rectangular prism or a cube, the constant of proportionality is $k = 1$, just as it is for rectangles and squares, so $V = lwh$. Is the constant of proportionality for a pyramid $\frac{1}{2}$, as it is for triangles, or is it a different value? We will find out on the next page.

Connect

Use a cube to derive the formula for the volume of a pyramid.

1 Find the volume formula of a cube.

A cube is a type of square prism. The volume of any prism is given by:

$V_{cube} = Bh$

The base is a square, so its area is the square of its side length, *s*. The height of a cube is this same distance, *s*.

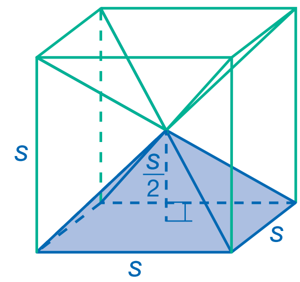
$V_{cube} = (s^2)(s) = s^3$

2 Divide a cube into 6 square pyramids.

Draw in the diagonals of the cube. Notice how the triangles formed by the diagonals and edges of the cube form 6 congruent pyramids. Each face of the cube is the base of a pyramid.

The volume of the cube is equal to the sum of the volumes of these 6 pyramids.

$V_{cube} = 6 \cdot V_{pyr}$



3 Consider one of these pyramids.

The volume of each pyramid must be proportional to its three dimensions.

$V_{pyr} = k_{pyr} \cdot l \cdot w \cdot h$

The length and width of each pyramid's base are equal to the side length of the cube, *s*. One pyramid stacked on top of another is equal to the height of the cube, so the height of each pyramid is half the height of the cube.

$V_{pyr} = k_{pyr} \cdot s \cdot s \cdot \frac{s}{2} = k_{pyr} \cdot \frac{s^3}{2}$

4 Find the general formula for the volume of a pyramid.

Find the constant of proportionality by substituting into the equation below.

$V_{cube} = 6 \cdot V_{pyr}$

$(s^3) = 6 \cdot \left(k_{pyr} \cdot \frac{s^3}{2}\right)$

$s^3 = 3 \cdot k_{pyr} \cdot s^3$

$1 = 3 \cdot k_{pyr}$

$\frac{1}{3} = k_{pyr}$

The volume of a square pyramid is given by:

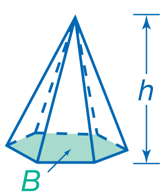
$V_{sqpyr} = \frac{1}{3}lwh$

More generally, the volume of a pyramid is proportional to the area of its base times its height.

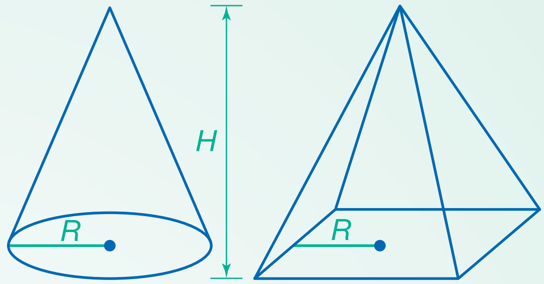
▶ $V_{pyr} = \frac{1}{3}Bh$

DISCUSS

Could you use the formula $V = \frac{1}{3}Bh$ to find the volume of this pyramid? Could you use $V = \frac{1}{3}lwh$? Explain.

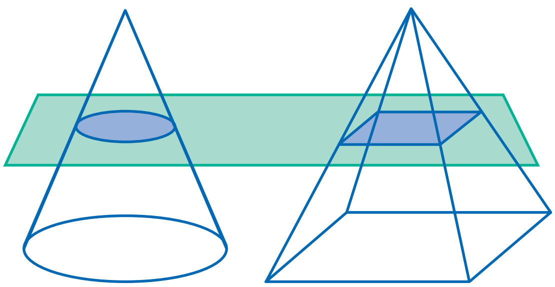


EXAMPLE A square pyramid and a cone have the same height, H. The base of the cone is a circle with radius R. This circle could be inscribed in the base of the pyramid, so the square base of the pyramid has an apothem of length R. Use these solids to derive the formula for the volume of a cone.



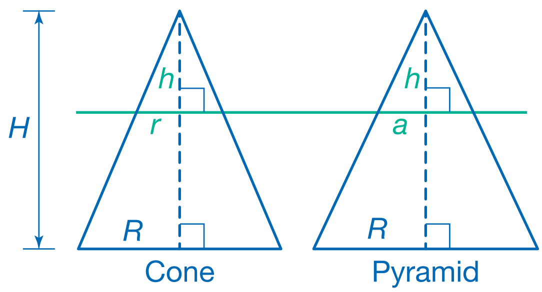
1 Take a cross section of each figure.

Slice a plane through the cone and the pyramid parallel to each base and at the same height.



2 Examine the figures from the front.

The cross section of the cone is a circle. Let its radius be r. The cross section of the pyramid is a square. Let its apothem be a.



3 Compare radius r to apothem a.

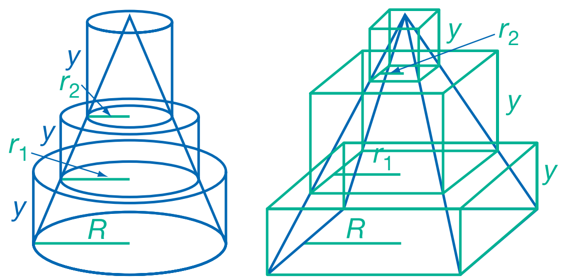
Since the plane is parallel to the base, the small triangles above the plane line are similar to the large triangles. Corresponding sides are proportional.

$\frac{r}{R} = \frac{h}{H}$ $\quad$ $\frac{a}{R} = \frac{h}{H}$

Because R, H, and h are the same for each figure, $r = a$, meaning the radius of the circular cross section is equal to the apothem of the square cross section.

4 Approximate the figures with cylinders and prisms.

Imagine taking several cross sections of the figures and then using those cross sections as the bases of short cylinders and prisms. These cylinders and prisms could be stacked to form figures like the cone and the pyramid.



5

Find the volumes of the stacked cylinders and prisms.

The volume of one cylinder in the stacked figure is given by $V_{cyl} = \pi r^2 y$.

The volume of the corresponding prism is given by $V_{pri} = (2r)^2 y = 4r^2 y$.

Relate these two volumes by finding their ratio.

$$\frac{V_{cyl}}{V_{pri}} = \frac{\pi r^2 y}{4r^2 y}$$

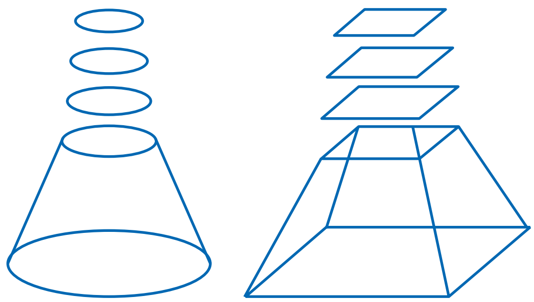
$$\frac{V_{cyl}}{V_{pri}} = \frac{\pi}{4}$$

$$V_{cyl} = \frac{\pi}{4} \cdot V_{pri}$$

6

Determine a formula for the volume of the cone.

If the height, y, of the cylinders and prisms is decreased, more are needed to build the figures. As the cylinders become thinner, the approximations come closer to the real volume of the figures. This process becomes like stacking circles and squares to build the figures.



$$V_{cone} = V_{cyl1} + V_{cyl2} + V_{cyl3} + V_{cyl4} + \ldots$$

$$V_{pyr} = V_{pri1} + V_{pri2} + V_{pri3} + V_{pri4} + \ldots$$

Recall that for any corresponding cylinder and prism in the "stacks," $V_{cyl} = \frac{\pi}{4}V_{pri}$.

$$V_{cone} = V_{cyl1} + V_{cyl2} + V_{cyl3} + V_{cyl4} + \ldots$$

$$V_{cone} = \frac{\pi}{4}V_{pri1} + \frac{\pi}{4}V_{pri2} + \frac{\pi}{4}V_{pri3} + \frac{\pi}{4}V_{pri4} + \ldots$$

$$V_{cone} = \frac{\pi}{4}(V_{pri1} + V_{pri2} + V_{pri3} + V_{pri4} + \ldots)$$

$$V_{cone} = \frac{\pi}{4}(V_{pri})$$

Insert the formula for the volume of a prism.

$$V_{cone} = \frac{\pi}{4}\left(\frac{1}{3}BH\right)$$

$$V_{cone} = \frac{\pi}{4}\left(\frac{1}{3}(2R)^2H\right) = \frac{\pi}{4}\left(\frac{1}{3}(4R^2)H\right)$$

$$V_{cone} = \frac{1}{3}\pi R^2 H$$

TRY

Ling is selling snow cones at a carnival. She uses cone-shaped paper cups that are 4 inches deep and 3 inches wide. She molds the top of each snow cone into a half-sphere. What is the volume of each snow cone?

Practice

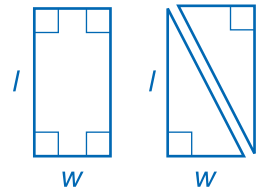
Identify the value of *k*, the constant of proportionality.

1. The rectangle is divided into two congruent triangles.

$A_{rect} = l \cdot w$

$A_{tri} = k_{tri} \cdot l \cdot w$

$k_{tri} =$ ________

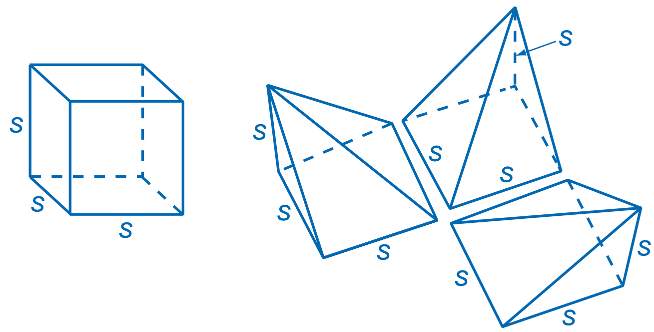


The images below show a cube and the cube cut into three congruent pyramids. Use these diagrams for questions 2–4.

2. Compare the base area and height of each oblique pyramid to the dimensions of the cube.

____

____



3. The cube has a volume of $V = s \cdot s \cdot s$. Each oblique pyramid has a volume of $V = k_{pyr} \cdot s \cdot s \cdot s$. What is the value of k_{pyr}? Explain how you know.

____

____

____

HINT Count the number of figures formed by dividing the cube.

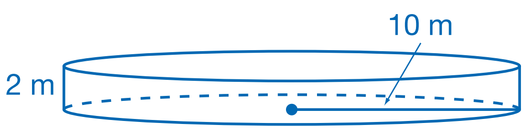
4. A right pyramid is one in which the apex is directly above the center of the base. The pyramids shown are oblique pyramids, because their apexes are not above the center of the base. Is the formula for finding the volume of an oblique pyramid the same as the formula for finding a right pyramid? Explain why this is or is not the case.

____

____

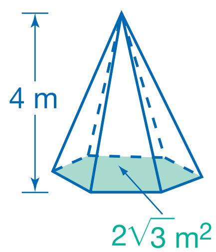
Find the volume of each solid figure shown. Give an exact answer.

5.



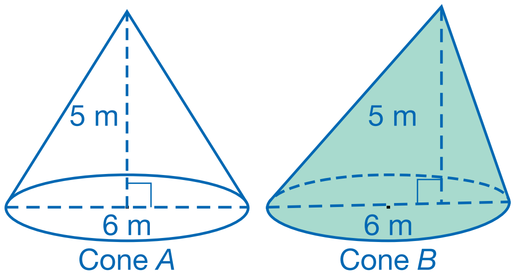
____

6.



____

Use the cones below for questions 7 and 8.



7. Find the volume of each cone above. Give exact answers.

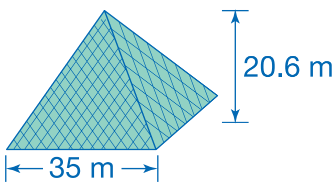
$V_A =$ ________

$V_B =$ ________

8. If the cones are cut by a horizontal plane parallel to their bases, will the areas of their cross sections be the same or different? Why? If different, which will have the greater area?

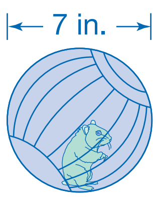
Solve. Give answers to the nearest tenth.

9. The entrance to the Louvre Museum in Paris, France, is a pyramid with a height of 20.6 meters and a square base with sides measuring 35 meters. What is the volume of the pyramid?



---

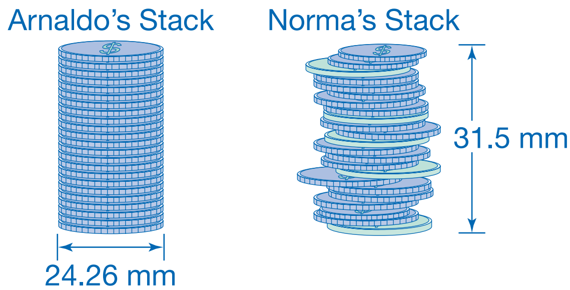
10. Leah bought a plastic ball for her hamster. The ball has a diameter of 7 inches. What is the volume of the ball?



Use the diagrams to the right for questions 11 and 12.

11. Arnaldo stacked 18 quarters one on top of another. Approximately how many cubic centimeters of metal were used to make all the quarters in Arnaldo's stack? Give your answer to the nearest tenth of a cubic centimeter.

____



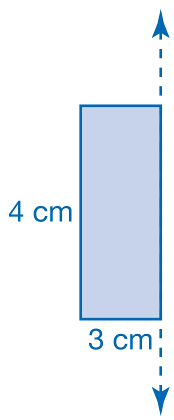
12. Norma made a stack of quarters, nickels, and pennies. She says the volume of her stack must be the same as the volume of Arnaldo's stack because the total height of her stack is the same as his. Is she correct? Explain your reasoning.

____

____

Find the volume of the figures below in terms of π.

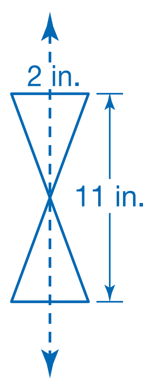
13. The figure below was rotated 360° around the dashed line to form a solid.



Find the volume of the solid formed. ______________________________

14. The figure on the right is made of congruent isosceles triangles. It was rotated 360° around the dashed line to form a solid.

Find the volume of the solid formed. ______________________________



Choose the best answer.

15. A cylindrical fish tank has a base radius of 7 inches. The volume of the tank is approximately 3,080 cubic inches. What is the approximate height of the fish tank?

A. 62 in. **C.** 11 in.

B. 20 in. **D.** 10 in.

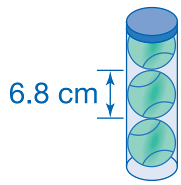
16. The volume of a small ice cream cone made by a company is 16π cubic centimeters. If the company doubles the base radius of the ice cream cone, what will be the volume of the new cone?

A. $10.\overline{6}\pi$ cm^3 **C.** 64π cm^3

B. 32π cm^3 **D.** 128π cm^3

Solve.

17. APPLY Three tennis balls, each with diameter 6.8 centimeters, fit in their container so that the top tennis ball touches the lid of the container and each tennis ball touches the sides of the container. The part of the container not taken up by the tennis balls contains air. How many cubic centimeters of this air are in the container? (Give your answer in terms of π.)



18. ANALYZE A furniture company sells sculptures in the shape of an oblique cone. Each sculpture has a base diameter of 6 inches and a height of 20 inches. The sculpture's apex lies directly above one of the points on the circular boundary of the base.

The company packs the sculptures in cylindrical containers with diameter and height equal to that of the sculpture. Two sculptures fit in each container. Do the sculptures completely fill the container? If not, what volume of air is in the packed container?