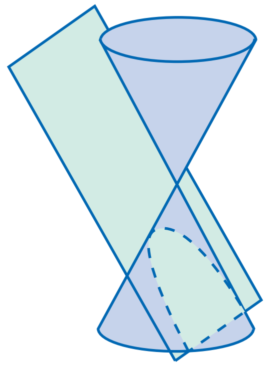
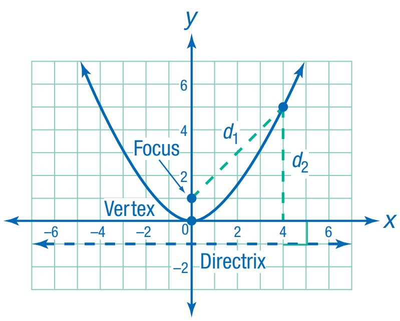
LESSON 22 Equations of Parabolas

UNDERSTAND A conic section is a two-dimensional cross section formed by the intersection of a plane and a double cone. When the plane intersects at an angle and passes through one of the bases, the cross section is a U-shaped curve called a **parabola**.

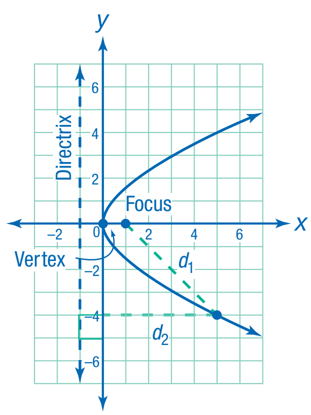


A conic section can also be defined as a curve having an equation of degree 2—a quadratic equation. The general form for the equation of any conic section is $Ax^2 + By^2 + Cx + Dy + E = 0$, in which A and B cannot both equal zero. If $B = 0$, the equation can be written in the form $y = ax^2 + bx + c$. This describes a parabola that opens up or down. If $A = 0$, the equation can be written in the form $x = ay^2 + by + c$. This describes a parabola that opens right or left.

A parabola is formally defined as the set of all points that are equidistant from a fixed line, called the **directrix**, and from a fixed point not on the line, called the **focus**. The parabola has an axis of symmetry that passes through the focus and is perpendicular to the directrix. The **vertex** lies on this line halfway between the focus and the directrix.



$d_1 = d_2$



The standard forms of an equation for a parabolic conic section with vertex (h, k) are given in the table below. The coordinates (h, k) give the vertex of the parabola. The variable p represents the shortest distance from the vertex to the focus and to the directrix. So, the coordinates of the focus and the equation of the directrix can be found from these equations.

	Vertical Parabola	Horizontal Parabola
Equation	$y - k = \frac{1}{4p}(x - h)^2$	$x - h = \frac{1}{4p}(y - k)^2$
Vertex	(h, k)	(h, k)
Focus	$(h, k + p)$	$(h + p, k)$
Directrix	$y = k - p$ (horizontal directrix)	$x = h - p$ (vertical directrix)

Connect

Find the vertex, focus, and directrix of the parabola represented by $x^2 + 2x - 16y + 33 = 0$

1 Put the equation in standard form.

The equation contains an x^2 term. So, begin by grouping only terms containing x on one side of the equation. Then apply the method of completing the square: compare the expression to $ax^2 + bx + c$ and add $\left(\frac{b}{2a}\right)^2$ to both sides of the equation. Then factor.

$$x^2 + 2x - 16y + 33 = 0$$
$$x^2 + 2x = 16y - 33$$
$$x^2 + 2x + \left(\frac{2}{2(1)}\right)^2 = 16y - 33 + \left(\frac{2}{2(1)}\right)^2$$
$$x^2 + 2x + 1 = 16y - 33 + 1$$
$$(x + 1)^2 = 16y - 32$$
$$\frac{1}{16}(x + 1)^2 = y - 2$$

2 Identify the vertex.

Compare the given equation to the standard form of a vertical parabola, $y - k = \frac{1}{4p}(x - h)^2$.

$$y - 2 = \frac{1}{16}(x + 1)^2$$
$$y - 2 = \frac{1}{16}(x - (-1))^2$$

We can see that $k = 2$ and $h = -1$.

▶ The vertex of the parabola, (h, k), is $(-1, 2)$.

3 Find the value of p.

To find the value of p, set $\frac{1}{4p}$ equal to the number in front of the squared binomial, $(x + 1)^2$.

$$\frac{1}{4p} = \frac{1}{16}$$
$$4p = 16$$
$$p = 4$$

4 Identify the focus and directrix.

This equation represents a vertical parabola. Its focus is given by $(h, k + p)$.

$(-1, 2 + 4) = (-1, 6)$

Its directrix is given by $y = k - p$.

$y = 2 - 4 = -2$

▶ The focus of the parabola is $(-1, 6)$ and the directrix is $y = -2$.

TRY

Find the vertex, focus, and directrix of $y^2 + x - 8y + 18 = 0$.

EXAMPLE A The focus of a parabola is (3.75, 5), and its directrix is $x = 4.25$. Write the equation of the parabola. Describe its appearance.

1 Determine the orientation of the parabola.

The directrix is a vertical line. So, the parabola is horizontal. Its equation has the form $x - h = \frac{1}{4p}(y - k)^2$.

2 Find the coordinates of the vertex.

Because the parabola is horizontal, the vertex is the midpoint between the focus (3.75, 5) and the point on the directrix with the same y-coordinate, (4.25, 5).

$$\left(\frac{3.75 + 4.25}{2}, \frac{5 + 5}{2}\right) = (4, 5)$$

The vertex is located at (4, 5).

3 Find the value of p.

The vertex (h, k) is (4, 5), so $h = 4$ and $k = 5$.

Because the parabola is horizontal, the formula for the focus is $(h + p, k)$. The focus of this parabola is (3.75, 5).

$$h + p = 3.75$$
$$4 + p = 3.75$$
$$p = -0.25 \text{ or } -\frac{1}{4}$$

This means the focus is $\frac{1}{4}$ unit to the left of the vertex.

4 Substitute the values of h, k, and p into the general form of the equation.

$h = 4$, $k = 5$, and $p = -0.25$

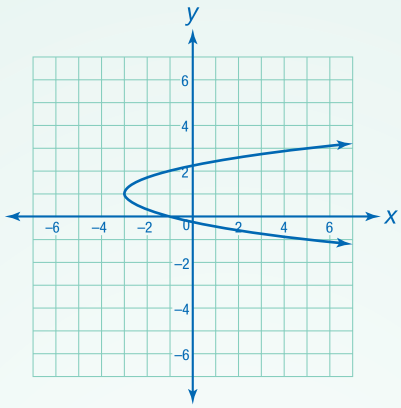
$$x - h = \frac{1}{4p}(y - k)^2$$
$$x - 4 = \frac{1}{4(-0.25)}(y - 5)^2$$
$$x - 4 = \frac{1}{-1}(y - 5)^2$$

▶ The equation of the parabola is $x - 4 = -(y - 5)^2$. Because the parabola is horizontal and $p < 0$, the parabola opens left.

MODEL

Graph the parabola. The point (0, 7) is on the parabola. What is the distance between this point and the focus, (3.75, 5)? What is the shortest distance between this point and the directrix, $x = 4.25$?

EXAMPLE B Identify the focus and the directrix for the parabola graphed below.



1 Identify the vertex and one other point.

The graph is a horizontal parabola that opens right. The vertex, or turning point, is the point with the lowest x-coordinate. The leftmost point appears to be $(-3, 1)$.

The point $(-1, 0)$ also appears to be on the graph.

2 Find the value of p.

The standard form equation of a horizontal parabola is $x - h = \frac{1}{4p}(y - k)^2$. Substitute the values from the vertex, $(-3, 1)$.

$x - h = \frac{1}{4p}(y - k)^2$

$x + 3 = \frac{1}{4p}(y - 1)^2$

Substitute $(-1, 0)$ into the equation and solve for p.

$$-1 + 3 = \frac{1}{4p}(0 - 1)^2$$

$$2 = \frac{1}{4p}(-1)^2$$

$$2 = \frac{1}{4p}(1)$$

$$8p = 1$$

$$p = \frac{1}{8}$$

3 Identify the focus and the directrix.

Because the parabola opens right, the focus is to the right of the vertex.

The formula for the focus is $(h + p, k)$.

$\left(-3 + \frac{1}{8}, 1\right) = \left(-\frac{23}{8}, 1\right)$

The directrix will be vertical and lie to the left of the parabola.

The formula for the directrix is $x = h - p$.

$x = -3 - \frac{1}{8} = -\frac{23}{8}$

▶ The focus is $\left(-\frac{25}{8}, 1\right)$, and the directrix is $x = -\frac{25}{8}$.

TRY

Write the equation of the parabola in the form $x = ay^2 + by + c$.

Practice

For each equation of a parabola, identify the vertex and the distance between the vertex and the focus, *p*.

1. $y + 10 = 2(x - 5)^2$

vertex: (_____, _____)

$p =$ _____

2. $x = -3(y - 1)^2 + 4$

vertex: (_____, _____)

$p =$ _____

3. $x - 2 = \frac{1}{4}(y + 8)^2$

vertex: (_____, _____)

$p =$ _____

Use the method of completing the square to rewrite each equation in standard form $y - k = \frac{1}{4p}(x - h)^2$ or $x - h = \frac{1}{4p}(y - k)^2$.

4. $x = y^2 + 6y + 10$

____

5. $x = 2y^2 + 4y - 3$

____

6. $\frac{1}{2}x^2 - y - 6x + 20 = 0$

____

HINT Factor out the coefficient 2 before completing the square.

Identify the focus and directrix for the parabola represented by each equation.

7. $y + 3 = -\frac{1}{12}x^2$

focus: (_____, _____)

directrix: _____

8. $x - 1 = \frac{1}{8}(y - 2)^2$

focus: (_____, _____)

directrix: _____

9. $y = x^2 - 10x + 27$

focus: (_____, _____)

directrix: _____

REMEMBER For a horizontal parabola, the focus is $(h + p, k)$ and the directrix is $x = h - p$.

Write the equation for the parabola with the given focus and directrix. Also identify the vertex and the direction (*up*, *down*, *left*, or *right*) in which the parabola opens.

10. focus: $(0, -2)$

directrix: $y = 2$

vertex: (_____, _____)

equation: __________

opens __________

11. focus: $(4, -4)$

directrix: $x = -2$

vertex: (_____, _____)

equation: __________

opens __________

12. focus: $\left(\frac{3}{2}, 3\right)$

directrix: $x = 5$

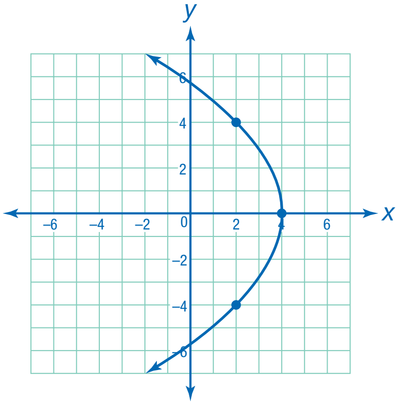
vertex: (_____, _____)

equation: __________

opens __________

Write an equation in standard form to represent each parabola. Identify the focus and directrix.

13.

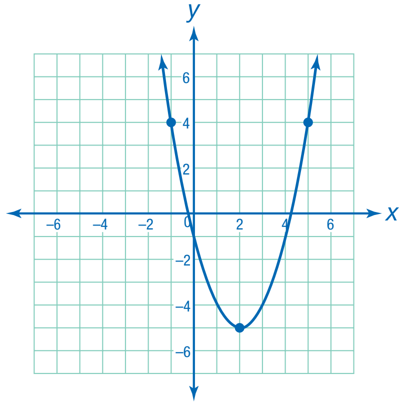


equation: ________________

focus: (______, ______)

directrix: ________________

14.



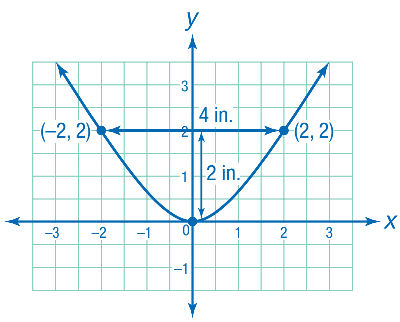
equation: ________________

focus: (______, ______)

directrix: ________________

Solve.

15. MODEL The mirrored reflector in a flashlight is shaped like a parabola. The reflector is 4 inches across and 2 inches deep. The lightbulb of the flashlight is located at the focus of the parabola. Label the location of the lightbulb on the diagram. How far from the vertex is the lightbulb positioned?



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