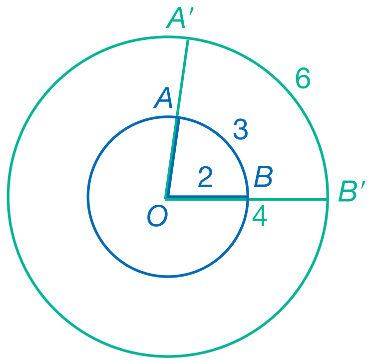
LESSON 26

Arc Lengths and Areas of Sectors

Arc Lengths and Radians



UNDERSTAND The previous lesson focused on the measure of an arc in degrees. Another way to measure an arc is to find the length along its curve in a unit of distance, such as inches or centimeters. Blue circle *O* has a radius of 2 units and contains arc *AB*, which has a length of 3 units.

Dilating the circle by a factor of 2 produces a circle with similar arc *A′B′*. Arcs *AB* and *A′B′* have the same measure because they have the same central angle, but they have different lengths. Because the scale factor of the dilation was 2, the length of the radius was doubled and the length of the arc was doubled.

For similar arcs (arcs with the same measure or the same central angle), a proportion can be set up by comparing arc length, *s*, and radius, *r*.

$$\frac{s_1}{r_1} = \frac{s_2}{r_2}$$

Circumference is the distance around a circle. The circumference of a circle is directly proportional to its diameter, *d*. For example, tripling the diameter of a circle also triples its circumference. The constant of proportionality that relates circumference to diameter is π. So, the formula for finding the circumference, *C*, of a circle is $C = \pi d$.

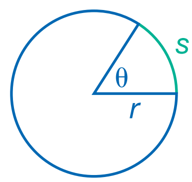
The ratio of the length of an arc, *s*, to the circumference of its circle, *C*, is equal to the ratio of the measure of the central angle, $t°$, to the full measure of the circle, 360°.

$$\frac{s}{C} = \frac{t°}{360°}$$

Recall that a diameter is twice the radius, $d = 2r$, and circumference is π times diameter, $C = \pi d$, so $C = 2\pi r$. Substitute this expression for *C* into the above formula.

$$\frac{s}{2\pi r} = \frac{t°}{360°} \qquad \text{Solve for } \frac{s}{r}.$$

$$\frac{s}{r} = t° \cdot \frac{2\pi}{360°}$$



Because 2, π, and 360° are all constants, we can define a new variable, θ (the Greek letter theta), to measure the central angle. Let $\theta = t° \cdot \frac{2\pi}{360°}$.

$$\frac{s}{r} = \theta \quad \rightarrow \quad s = \theta r$$

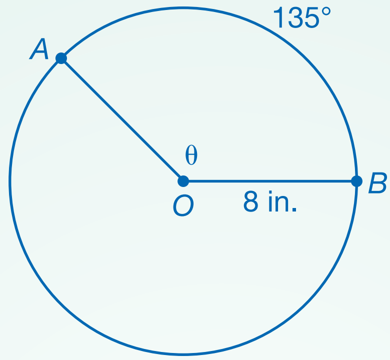
θ represents an angle measure in **radians**. Like degrees, radians are units for measuring angles. A full circle (360°) contains 2π radians.

$$\theta_{\text{circle}} = t°_{\text{circle}} \cdot \frac{2\pi}{360°} = 360° \cdot \frac{2\pi}{360°} = 2\pi$$

180° is equivalent to π radians, and a right angle (a 90° angle) contains $\frac{\pi}{2}$ radians.

Connect

What is the length of $\overset{\frown}{AB}$ in circle O?



1 Find the measure of angle AOB.

Angle AOB is a central angle that intercepts $\overset{\frown}{AB}$.

Since $m\overset{\frown}{AB} = 135°$, $m\angle AOB = 135°$.

2 Convert the measure of angle AOB to radians.

360° is equivalent to 2π radians, so:

$\theta = 135° \cdot \frac{2\pi}{360°} = \frac{270}{360}\pi = \frac{3}{4}\pi$

3 Find the arc length, s.

$s = \theta r$

$s = \left(\frac{3}{4}\pi\right)(8)$

$s = 6\pi$

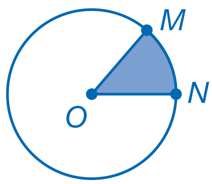
▶ Arc AB measures 6π inches, or approximately 18.84 inches.

CHECK

Solve the problem by using the relation $\frac{s}{C} = \frac{t°}{360°}$. Compare the calculations you did to the work above.

Sectors

UNDERSTAND A **sector** of a circle is the region bound by two radii and the arc that connects them. The shaded region of circle O shows sector MON.



Just as the length of an arc is a portion of the circle's circumference, the area of a sector is a portion of the circle's total area. Recall that the formula for the area of a circle is $A_{\text{circle}} = \pi r^2$. If the central angle and the arc of a sector have a degree measure of $t°$, then this proportion can be used to find A_{sector}, the area of the sector:

$$\frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{t°}{360°}$$

$$\frac{A_{\text{sector}}}{\pi r^2} = \frac{t}{360}$$

$$A_{\text{sector}} = \frac{t}{360} \cdot \pi r^2$$

UNDERSTAND The area of a sector can also be found by using θ, the measure of its central angle in radians. Begin by setting up a proportion to relate $t°$ and θ. A circle contains 360°, or 2π radians.

$$\frac{t°}{360°} = \frac{\theta}{2\pi}$$

Substitute the expression on the right in the above equation into the relation for the area of a sector. Then simplify.

$$A_{\text{sector}} = \frac{\theta}{2\pi} \cdot \pi r^2$$

$$A_{\text{sector}} = \frac{1}{2}\theta r^2$$

UNDERSTAND Recall that θ represents the ratio of an arc length to the radius of its associated circle, $\theta = \frac{s}{r}$. This expression can be substituted into the formula above.

$$A_{\text{sector}} = \frac{1}{2}\theta r^2$$

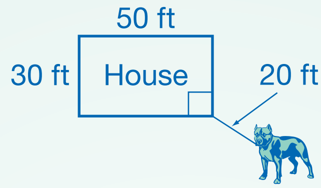
$$A_{\text{sector}} = \frac{1}{2}\left(\frac{s}{r}\right) r^2$$

$$A_{\text{sector}} = \frac{1}{2}sr$$

The new formula allows the area of a sector to be calculated if the lengths of the radii and arc that enclose it are known.

Connect

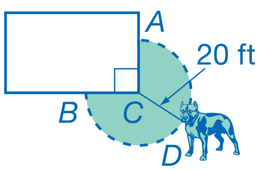
A dog is leashed to the corner of a rectangular house, as shown. Approximately how much area does the dog have in which to run?



1 Sketch the area in which the dog can run.

The dog can wander up to 20 feet along either adjacent wall. Since both walls are at least 20 feet long, the dog cannot go around any other corners.

The area in which the dog can run is a sector of a circle. The length of the leash is the circle's radius.



2 Find the measure of arc *BDA*.

Angle *BCA* is a 90° angle.

So, the arc of the circle that is covered by the house measures 90°.

The measure of $\overset{\frown}{BDA}$ must, therefore, be:

$360° - 90° = 270°$

3 Find the area in which the dog can run.

$A = \frac{m\overset{\frown}{BDA}}{360°} \cdot \pi r^2$

$A = \frac{270}{360} \cdot \pi(20)^2$

$A = \frac{3}{4} \cdot 400\pi$

$A = 300\pi$

$A \approx 300 \cdot 3.14 \approx 942 \text{ ft}^2$

▶ The area in which the dog can run is approximately 942 square feet.

CHECK

Convert the degree measure of the central angle of the sector to radians. Then use radians to determine the area. Is the result the same?

Practice

Complete each table by converting the given measure to its equivalent measure in degrees or radians.

1.

Degrees	Radians
0°	
30°	
	$\frac{\pi}{4}$
	$\frac{\pi}{2}$

2.

Degrees	Radians
	$\frac{2\pi}{3}$
	π
270°	
360°	2π

REMEMBER $360° = 2\pi$ rad

Write an appropriate word to complete each statement.

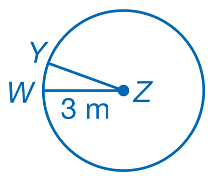
3. Angles and arcs can be measured in degrees or in ____________.

4. The length of a(n) ____________ is a fraction of the circumference of a circle.

5. A sector is a region of a circle bounded by a(n) ______ and two ____________ of the circle.

Choose the best answer.

6. What is the area of sector *WZY* if the measure of $\angle WZY$ is $\frac{\pi}{9}$ radians?



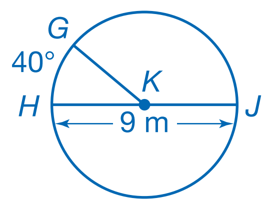
A. $\frac{\pi}{3}$ m^2

B. $\frac{\pi}{2}$ m^2

C. 2π m^2

D. 3π m^2

7. What is the length of $\overset{\frown}{GJ}$, in radians, if m$\overset{\frown}{GH}$ = 40° and $\overline{HJ}$ is a diameter?



A. π m

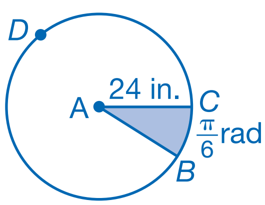
B. 2π m

C. $\frac{7\pi}{9}$ m

D. 7π m

Find each indicated arc length and sector area in each circle *A*.

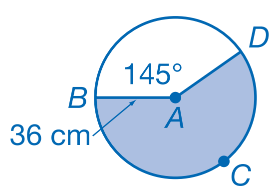
8.



length of $\overset{\frown}{CB}$ = ______

area of shaded sector = ______

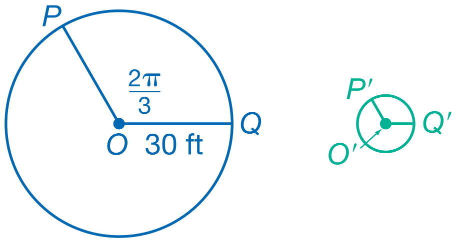
9.



length of $\overset{\frown}{BCD}$ = ______

area of shaded sector = ______

Circle O was dilated by a scale factor of $\frac{1}{3}$ and translated to the right to form circle O'. Use these circles for questions 10–12.



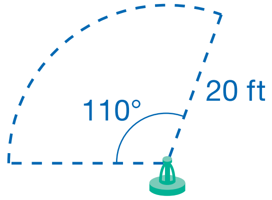
10. How does the radian measure of $\overset{\frown}{PQ}$ compare to the measure of $\overset{\frown}{P'Q'}$?

11. What are the lengths of $\overset{\frown}{PQ}$ and $\overset{\frown}{P'Q'}$? How do they compare to each other?

12. What are the areas of sectors POQ and $P'O'Q'$? How do they compare to each other?

Solve.

13. **APPLY** A lawn sprinkler is set to spray water over a distance of 20 feet and rotate through an angle of 110°. What is the approximate area of the lawn that will be watered? Explain how you found your answer.



14. **COMPUTE** Line segments OM and ON are radii of circle O. The shaded area that is bounded by $\overline{MN}$ and $\overset{\frown}{MN}$ is called a segment of the circle. Compute the approximate area of this segment.