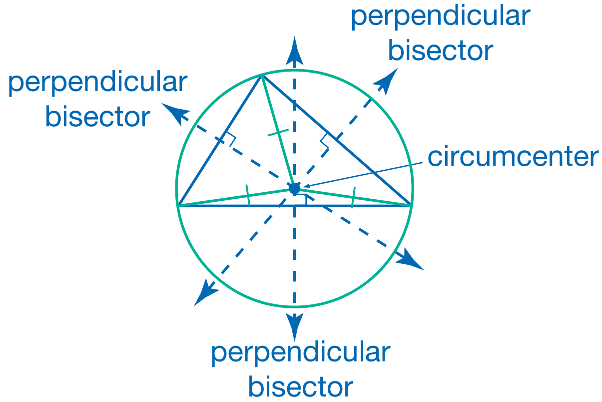
LESSON 27 Constructions with Circles

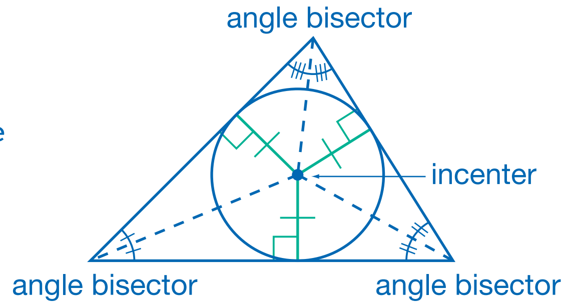
UNDERSTAND A **point of concurrency** is a point at which three or more lines intersect. Recall that a perpendicular bisector is a line that is perpendicular to a line segment and passes through the midpoint of that segment. A triangle has three sides. The point at which the three perpendicular bisectors of those sides intersect is called the **circumcenter**. As its name suggests, the circumcenter is the center of the **circumscribed circle** that fits around the triangle.



A circle is circumscribed about a triangle if all three of the triangle's vertices are on the circle. A line segment drawn from one of those vertices to the circumcenter is a radius of the circle. So, the circumcenter is equidistant from all three vertices of the triangle.

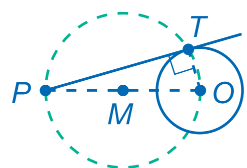
Recall that an angle bisector is a line or ray that divides an angle in half. A triangle contains three angles. The point at which the three angle bisectors of those angles intersect is the **incenter** of that triangle. As its name suggests, the incenter is the center of the **inscribed circle** that fits within the triangle.

A circle is inscribed in a triangle if every side of the triangle is tangent to the circle. The circle and the triangle intersect at exactly three points. A line segment drawn from one of those points of tangency to the incenter is a radius of the inscribed circle. So, the incenter is equidistant from all three sides of the triangle.



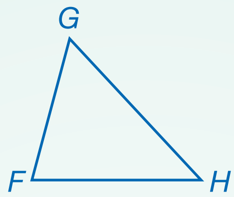
UNDERSTAND The circumcenter of a right triangle is always the midpoint of its hypotenuse. Notice that when a circle is circumscribed about a polygon, every angle in that polygon is an inscribed angle of the circumscribed circle. An inscribed right angle intercepts a semicircle, so the hypotenuse of the triangle must also be a diameter of the circumscribed circle.

In the diagram below, $\overleftrightarrow{PT}$ is tangent to circle O at point T, so $\overleftrightarrow{PT}$ must be perpendicular to radius $\overline{OT}$. Notice that these two line segments can be seen as the legs of right triangle PTO, which has $\overline{OP}$ as its hypotenuse. The midpoint of $\overline{OP}$ is the center of circle M, the circle that circumscribes $\triangle PTO$. This circle contains exterior point P, point of tangency T, and the center of the circle, O. So, if you did not know where the point of tangency was, you could find it by finding the midpoint of $\overline{PO}$ and then constructing the circle with that midpoint as its center and $\overline{PO}$ as its diameter.



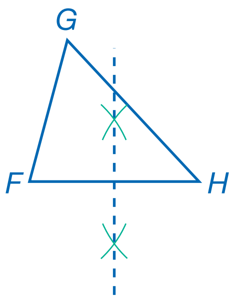
Connect

Circumscribe a circle about △*FGH*.



1 Construct the perpendicular bisector of $\overline{FH}$.

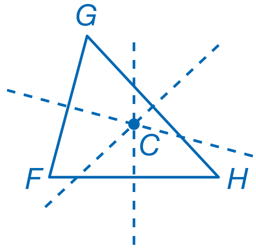
Use a compass to draw equivalent arcs from endpoints *F* and *H*. Identify the points where these arcs intersect and connect them to form the perpendicular bisector.



2 Find the circumcenter.

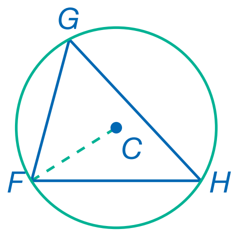
Use similar processes to construct the perpendicular bisectors of sides $\overline{FG}$ and $\overline{GH}$.

Label the circumcenter *C* at the point where the perpendicular bisectors intersect.



3 Draw the circumscribed circle.

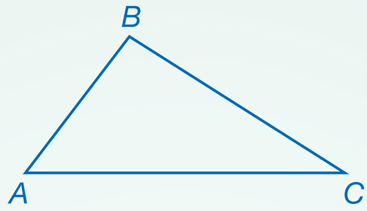
Place the compass point on *C* and the pencil point on *F*. Draw a circle with radius *CF*.



DISCUSS

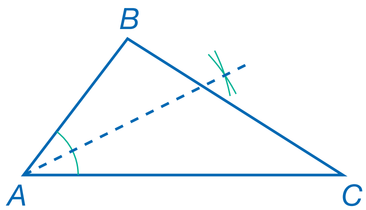
Could you have measured a different distance, other than *CF*, to draw the circumscribed circle? Explain.

EXAMPLE A Inscribe a circle inside $\triangle ABC$.



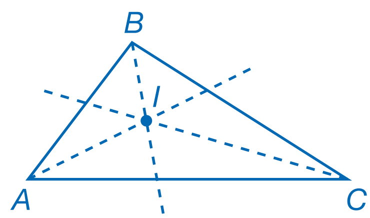
1 Construct the angle bisector of $\angle A$.

Draw an arc from angle A that intersects $\overline{AB}$ and $\overline{AC}$. From these intersection points, draw arcs to locate a point on the angle bisector.



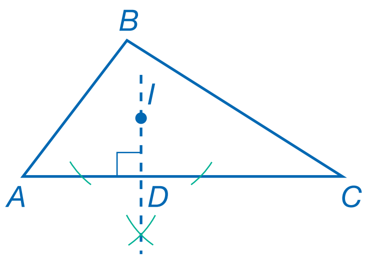
2 Find the incenter.

Use similar steps to construct the bisectors of angles B and C. Label the incenter I at the point where the angle bisectors intersect.



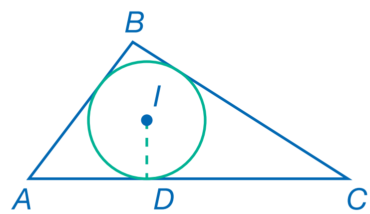
3 Find a radius of the inscribed circle.

Construct a line perpendicular to side $\overline{AC}$ passing through point *I*. From point *I*, draw an arc that intersects $\overline{AC}$. From the intersection points of the arc, draw arcs to locate a point on the perpendicular line. Label the point where the perpendicular line crosses $\overline{AC}$ as point *D*.



4 Draw the inscribed circle.

Place the compass point on *I* and the pencil point on *D*. Draw a circle with radius *ID*.



DISCUSS

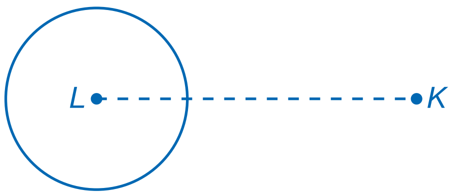
Did you need to draw all three angle bisectors to find the incenter of $\triangle ABC$?

EXAMPLE B Point *K* lies outside circle *L*. Draw two tangent lines to circle *L* from point *K*.

1

Draw a segment connecting point *K* to the center of the circle, point *L*.

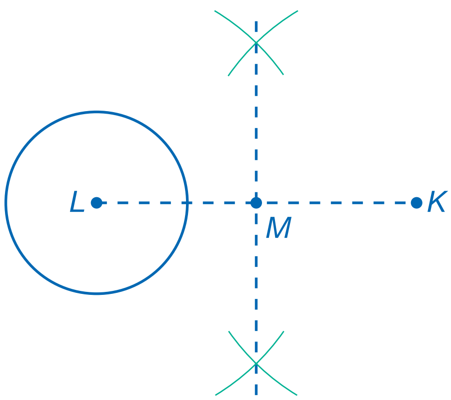
Use a straightedge to draw $\overline{LK}$.



2

Find the midpoint of $\overline{LK}$.

Construct the perpendicular bisector to find the midpoint. Label the midpoint *M*.

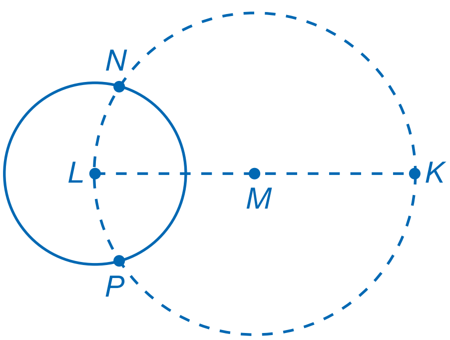


3

Draw the circle that has center *M* and radius *LM*.

Place the compass point on *M* and the pencil point on *L*. Draw a circle with radius *LM*.

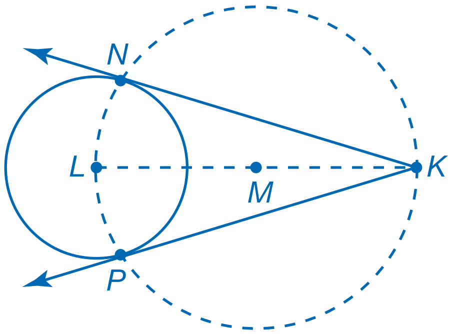
Label the points where circles *M* and L intersect as *N* and *P*.



4

Draw $\overleftrightarrow{KN}$ and $\overleftrightarrow{KP}$.

Use a straightedge to draw a line connecting point *K* to point *N* and another line connecting point *K* and point *P*.



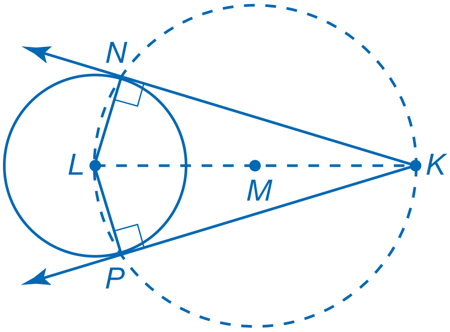
5

Check that $\overleftrightarrow{KN}$ and $\overleftrightarrow{KP}$ are tangent to circle *L*.

Draw in radius $\overline{LN}$. Angle *LNK* is an inscribed angle of circle *M*. Its intercepted arc is semicircle *LPK*, so $\angle LNK$ must be a right angle. Thus, $\overleftrightarrow{KN} \perp \overline{LN}$.

A tangent line is perpendicular to a circle at the point of tangency, so point *N* is the point at which $\overrightarrow{KN}$ is tangent to circle *L*.

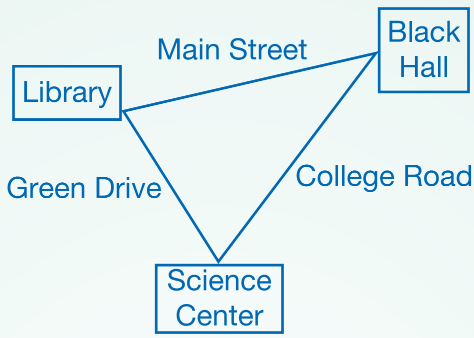
Draw in radius $\overline{LP}$. Angle *LPK* intercepts semicircle *LNK*, so the same reasoning can be used to show that $\overleftrightarrow{KP}$ is tangent to circle *L*.



DISCUSS

Is it possible to draw a third tangent line (other than $\overleftrightarrow{KN}$ and $\overleftrightarrow{KP}$) from point *K* to circle *L*? Explain.

EXAMPLE C The diagram below shows plans for three buildings on a college campus and the three roads connecting them. The planning committee wishes to place a water fountain in the center of campus, equidistant from each of the three buildings. Add the fountain to the plans.



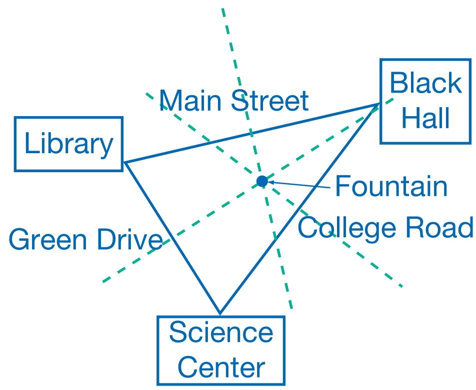
1 Examine the problem.

The fountain will lie on a point equidistant from the three buildings, each of which touches a vertex of the triangle.

The point equidistant from the three vertices of a triangle is the circumcenter.

2 Locate the circumcenter of the triangle.

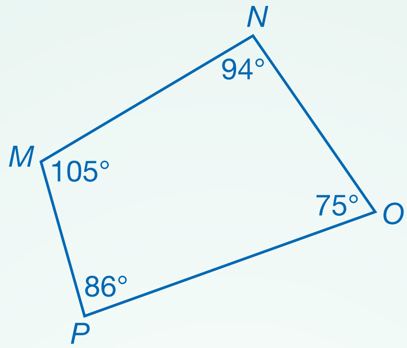
Construct the perpendicular bisectors. The fountain will be located where the three perpendicular bisectors intersect one another.



TRY

Suppose instead that the committee chose to place the fountain so that it is equidistant from the three roads. Mark this location on the diagram.

EXAMPLE D If possible, circumscribe a circle around quadrilateral *MNOP*.



1 Determine if the figure can be circumscribed by a circle.

A quadrilateral can be circumscribed by a circle if its opposite angles are supplementary.

$m\angle M + m\angle O = 105° + 75° = 180°$ ✓

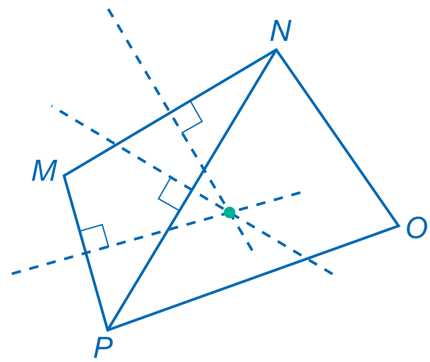
$m\angle N + m\angle P = 94° + 86° = 180°$ ✓

MNOP can be circumscribed by a circle.

2 Find the circumcenter of $\triangle MNP$.

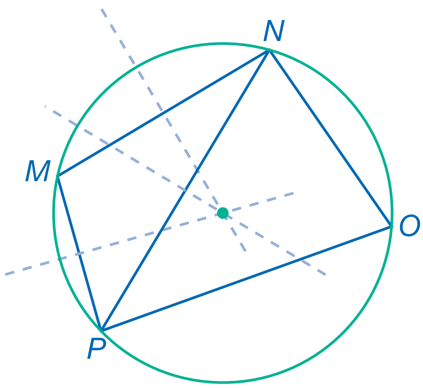
Drawing diagonal $\overline{NP}$ divides *MNOP* into two triangles. The circle that circumscribes those triangles also circumscribes the quadrilateral.

Construct the perpendicular bisectors of the sides of $\triangle MNP$ to locate its circumcenter.



3 Draw the circumscribed circle.

Place the compass point on the circumcenter and the pencil point on *M*. Draw a circle around the figure.



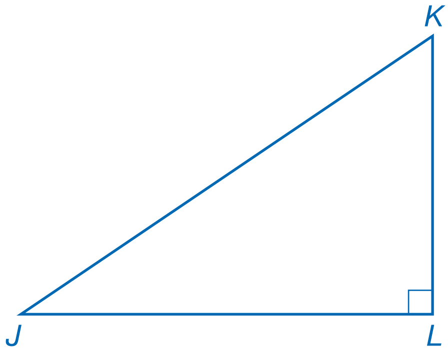
DISCUSS

Would the perpendicular bisectors of $\triangle NOP$ intersect at the same point? Could you find the center of the circumscribed circle by finding the perpendicular bisectors of the sides of *MNOP*, without dividing it into triangles?

Practice

Perform each construction.

1. Inscribe a circle in $\triangle JKL$. Identify the point of concurrency that is the center of the circle you drew.



point of concurrency:

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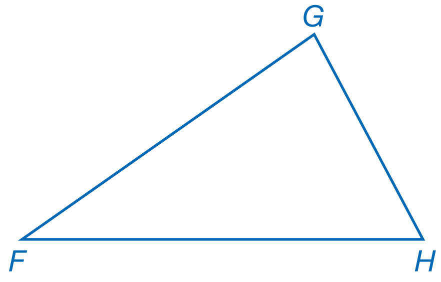
REMEMBER: To inscribe a circle in a polygon, you need to construct angle bisectors.

2. Circumscribe a circle about $\triangle FGH$. Identify the point of concurrency that is the center of the circle you drew.



point of concurrency:

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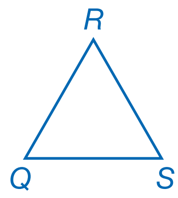
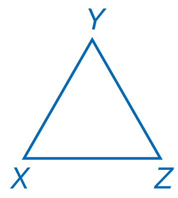
HINT The name of the point of concurrency is similar to the word *circumscribe*.

3. Point *A* is outside circle *C*. Construct two lines through point *A* that are tangent to circle *C*.

The triangles below are equilateral and congruent. Use them for questions 4–6.





4. Construct the circumcenter of $\triangle QRS$. Label it point *C*.

5. Construct the incenter of $\triangle XYZ$. Label it point *I*.

6. What do you notice about the circumcenter and incenter of the two triangles?

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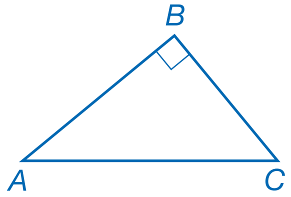
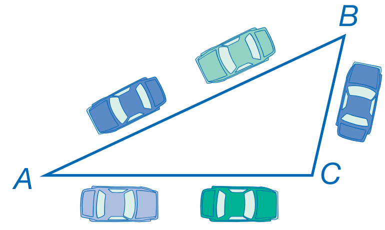
Perform the necessary constructions.

7. SHOW Construct the circumcenter of $\triangle ABC$ below to show that it is located at the midpoint of the hypotenuse.



8. DRAW Adam is talking on the phone in a triangular park that is surrounded on all sides by roads. The noise from the cars is so loud that he is having difficulty hearing. Draw a point to show the location where he should stand if he wants to be as far as possible from any of the traffic. Explain how you know that is the correct location.



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