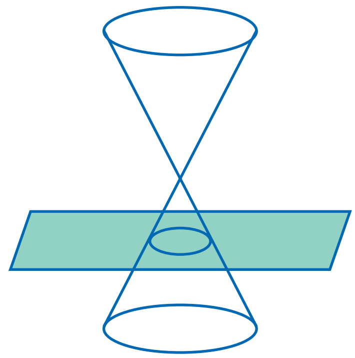
LESSON 28 Equations of Circles

UNDERSTAND Recall that a circle can be defined as a collection of points that are equidistant from a given point, called the center. A circle can also be defined as a type of conic section. Recall that a conic section is a two-dimensional cross section formed by the intersection of a plane and a double cone like the one shown. When the cone is sliced horizontally, by a plane parallel to its base, the cross section is a circle.



Recall that a conic section is a curve having degree 2, so it can be represented by a quadratic equation. The general form for the equation of a conic section is $Ax^2 + By^2 + Cx + Dy + E = 0$, in which A and B cannot both equal zero. When $A = B$, as in $4x^2 + 4y^2 + 4x - 12y + 6 = 0$, the equation represents a circle. An equation of a circle has two quadratic terms, such as $4x^2$ and $4y^2$. But, because the greatest exponent of any term is 2, the equation of a circle is classified as a quadratic equation.

UNDERSTAND The center of a circle is the point that is an equal distance from all points on the circle. That distance is the radius of the circle. When a circle is graphed on the coordinate plane, knowing the center and the radius allows you to determine its equation.

The standard form of the equation of a circle with center (h, k) and radius r is:

$$(x - h)^2 + (y - k)^2 = r^2$$

For example, the circle on the coordinate plane with its center at $(2, -3)$ and a radius of 5 units has the equation $(x - 2)^2 + (y + 3)^2 = 25$.

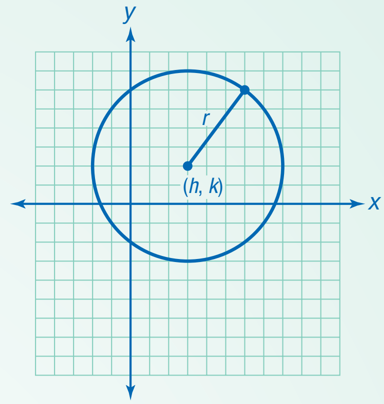
Every point (x, y) on a circle is a solution to the equation of that circle. The equation of a circle or other conic section allows the figure to be examined on the coordinate plane.

UNDERSTAND The center and radius of a circle can be determined, given its equation. Consider the equation $x^2 + y^2 = 9$. The length of the radius is equal to the principal square root of the constant term. So, if $r^2 = 9$, then $r = 3$. (The radius r cannot equal -3 because a length cannot be negative.) The equation can be rewritten to make the values of h, k, and r obvious: $(x - 0)^2 + (y - 0)^2 = 3^2$. So, the center is $(0, 0)$.

When the equation is given in general form or some other form, it is necessary to convert it into standard form in order to determine the center and radius. This can be done by completing the square for the terms containing x and the terms containing y.

Connect

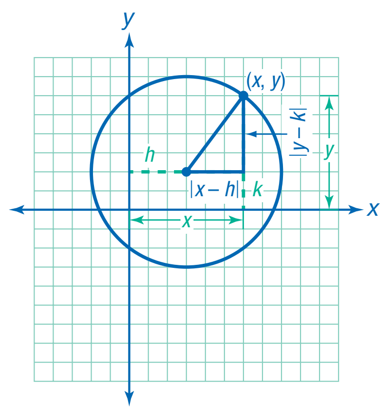
The circle on the coordinate plane on the right has center (h, k) and radius r. Use this circle to derive the equation of a circle.



1 Find the endpoints of the radius.

One endpoint of the radius is the center (h, k). Label the other endpoint, which is on the circle, (x, y). Notice that (x, y) could stand for any point on the circle.

2 Using the radius as the hypotenuse, draw a right triangle.



3 Find expressions for the lengths of the legs of the triangle.

Notice that the vertex of the right angle of the triangle is the point (x, k). The length of the vertical leg of the triangle is the distance from point (x, y) to this point, (x, k). Use the distance formula to find this length.

$$l_1 = \sqrt{(x - x)^2 + (y - k)^2}$$

$$l_1 = \sqrt{(0)^2 + (y - k)^2}$$

$$l_1 = \sqrt{(y - k)^2}$$

$$l_1 = |y - k|$$

The length of the horizontal leg of the triangle is the distance from point (x, k) to the center, (h, k).

$$l_1 = \sqrt{(x - h)^2 + (k - k)^2}$$

$$l_1 = |x - h|$$

4 Apply the Pythagorean Theorem

Substitute $|x - h|$ and $|y - k|$ for the leg lengths and r for the hypotenuse length.

$$a^2 + b^2 = c^2$$

$$|x - h|^2 + |y - k|^2 = r^2$$

Since those expressions are squared, the absolute value symbols can be dropped.

▶ $(x - h)^2 + (y - k)^2 = r^2$

TRY

Identify the center and radius of the circle represented by $(x - 4)^2 + (y + 3)^2 = 4$.

EXAMPLE A Convert $3x^2 + 3y^2 - 12x + 6y - 12 = 0$ to standard form. Then graph the circle.

1 Complete the square for the x terms.

Divide out the GCF, 3, and then group the x terms.

$$3x^2 + 3y^2 - 12x + 6y - 12 = 0$$
$$x^2 + y^2 - 4x + 2y - 4 = 0$$
$$(x^2 - 4x) + y^2 + 2y - 4 = 0$$

Compare $x^2 - 4x$ to $ax^2 + bx + c$.

$a = 1$ and $b = -4$, so $\left(\frac{b}{2a}\right)^2 = \left(\frac{4}{2(1)}\right)^2 = 4$.

Add 4 to both sides. Then factor the perfect square trinomial.

$$(x^2 - 4x + 4) + y^2 + 2y - 4 = 0 + 4$$
$$(x - 2)^2 + y^2 + 2y - 4 = 4$$

2 Complete the square for the y terms.

Group the y terms, and move all constant terms to the right side of the equation.

$$(x - 2)^2 + (y^2 + 2y) - 4 = 4$$
$$(x - 2)^2 + (y^2 + 2y) = 8$$

For the y-terms, $a = 1$ and $b = 2$, so $\left(\frac{b}{2a}\right)^2 = \left(\frac{2}{2(1)}\right)^2 = 1$. Add 1 to both sides. Then factor the trinomial.

$$(x - 2)^2 + (y^2 + 2y + 1) = 8 + 1$$
$$(x - 2)^2 + (y + 1)^2 = 9$$

3 Identify the center and radius.

Compare $(x - 2)^2 + (y + 1)^2 = 9$ to $(x - h)^2 + (y - k)^2 = r^2$.

$$r^2 = 9$$
$$r = 3$$

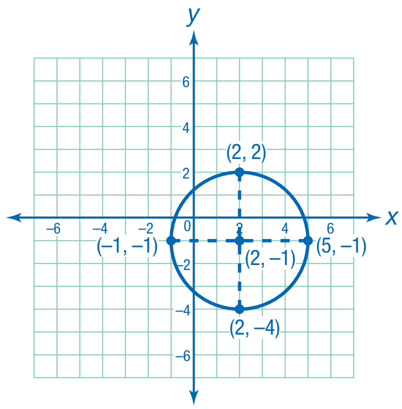
The radius of the circle is 3.

$x - h = x - 2$	$y - k = y + 1$
$-h = -2$	$-k = 1$
$h = 2$	$k = -1$

The center of the circle is $(2, -1)$.

4 Graph the circle.

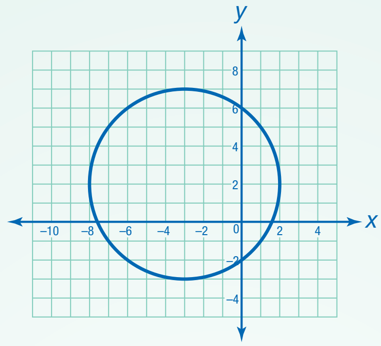
Plot the center $(2, -1)$. Then, count 3 units right, left, up, and down to find points on the circle—$(5, -1)$, $(-1, -1)$, $(2, 2)$, and $(2, -4)$. Connect the four points with a smooth curve.



CHECK

Substitute points on the circle into the general form of the equation to verify that the graph is correct.

EXAMPLE B A circle is graphed below. Write the equation for the circle in standard form.



1 Identify the radius of the circle.

Points $(-3, 7)$ and $(-3, -3)$ appear to lie on opposite ends of the circle. Calculate the distance between those points.

$d = \sqrt{(-3 - (-3))^2 + (-3 - 7)^2}$

$d = \sqrt{(0)^2 + (-10)^2}$

$d = \sqrt{100}$

$d = 10$

This can be verified by using two other points, such as $(-8, 3)$ and $(2, 3)$.

$d = \sqrt{(2 - (-8))^2 + (3 - 3)^2} = 10$

If $d = 10$ units, then $r = \frac{10}{2} = 5$ units.

2 Identify the center of the circle.

The center of the circle is the midpoint of the diameters. Find the average of the coordinates of the endpoints of a diameter, $(-3, 7)$ and $(-3, -3)$:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-3 + (-3)}{2}, \frac{7 + (-3)}{2}\right)$$
$$= (-3, 2)$$

3 Substitute the values you found into the standard form of the equation.

The center is $(-3, 2)$. The radius is 5.

$(x - h)^2 + (y - k)^2 = r^2$

$(x - (-3))^2 + (y - 2)^2 = 5^2$

▶ The equation of the circle in standard form is $(x + 3)^2 + (y - 2)^2 = 25$.

TRY

Write the general form of the equation for this circle.

Practice

Identify the center and radius of the circle described by each equation.

1. $(x-8)^2 + (y-1)^2 = 81$

center: (______, ______)

radius: ______

2. $(x+4)^2 + (y+1)^2 = 10$

center: (______, ______)

radius: ______

3. $(x-6)^2 + \left(y+\frac{4}{3}\right)^2 = \frac{64}{9}$

center: (______, ______)

radius: ______

REMEMBER The standard form $(x-h)^2 + (y-k)^2 = r^2$ describes a circle with center (h, k) and radius r.

Convert each equation of a circle from general form to standard form by completing the square. Then identify the circle's center and radius.

4. $x^2 + y^2 - 8x + 4y - 16 = 0$

standard form: ____________________

center: (______, ______)

radius: ______

5. $4x^2 + 4y^2 - 8y = 0$

standard form: ____________________

center: (______, ______)

radius: ______

HINT Remove a factor of 4 before completing the square.

Write an appropriate word or phrase to complete each sentence.

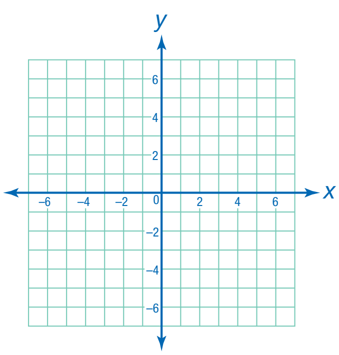
6. A conic section is a figure formed by the intersection of a plane and a(n) __________.

7. A conic section is a curve having an equation of degree ______.

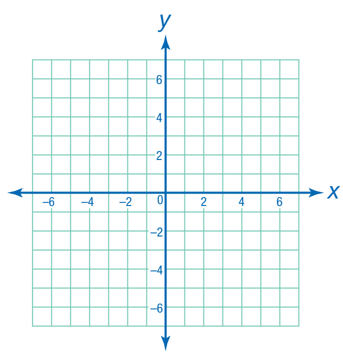
8. The general form for the equation of a circular conic section is $Ax^2 + By^2 + Cx + Dy + E = 0$, where ______ = ______.

Graph the circle that is represented by each equation.

9. $(x+1)^2 + (y-1)^2 = 9$

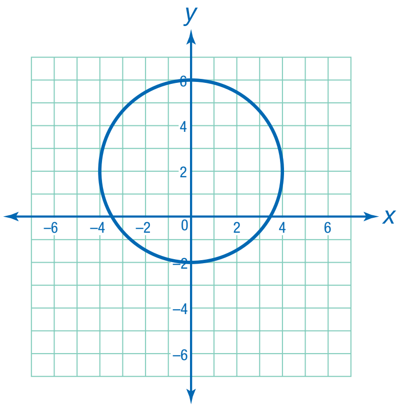


10. $x^2 + y^2 - 4x - 8y + 16 = 0$

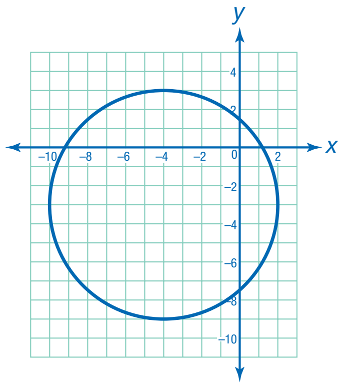


Write an equation to represent each circle in standard form. Then write the equation in the general form for a conic section.

11.



12.



Solve.

13. A circle has a diameter with endpoints at $(-3, -10.25)$ and $(-3, 1.75)$. What is the equation of the circle in standard form?

14. WRITE Point $(-7, 0)$ is on a circle with center $(-8, -4)$. Write the equation of the circle.

15. JUSTIFY The grid shows a map of an acoustic music festival. Each unit on the grid represents $\frac{1}{10}$ mile. The volunteer coordinator gives each volunteer a walkie-talkie with a range of $\frac{1}{2}$ mile. Will the coordinator be able to communicate from her kiosk with all sites shown on the map? Justify your answer.