LESSON 32

Permutations and Combinations

UNDERSTAND The **factorial** of an integer n is the product of all positive integers less than or equal to n. The factorial of a number n is written as $n!$.

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \ldots \cdot 2 \cdot 1 \qquad 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

Since factorials are made up of integer factors, you can divide factorials by cancelling common factors. You can simplify $\frac{9!}{7!}$ without calculating either factorial.

$$\frac{9!}{7!} = \frac{9 \cdot 8 \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 9 \cdot 8 = 72$$

UNDERSTAND A **permutation** is an arrangement of items or events in a particular order. A different order of the same items or events is a different permutation. A password, such as ABC123, is an example of a permutation. If you enter in the characters of a password in an incorrect order, such as A1B2C3, the password will not be accepted.

You can use factorials to find the number of possible permutations in a given situation. For example, in how many different orders can 4 students line up against a wall? When choosing a student for the first spot, you have 4 options. Once the first spot is filled there are 3 options remaining for the second spot, then 2 for the next spot, and then 1 for the last spot. The total is $4 \cdot 3 \cdot 2 \cdot 1$, or $4!$.

The formula for finding the number of permutations of n objects taken r at a time is:

$${}_nP_r = \frac{n!}{(n-r)!}$$

Suppose that a club with 20 members wants to select a president and secretary. The students could select John and Anne, or Anne and David. Also, selecting John for president and Anne for secretary is not the same as selecting Anne for president and John for secretary. There are 20 possible presidents. Once the president is chosen, there are 19 possibilities for secretary. In mathematical language, this is represented by ${}_{20}P_2$.

$${}_{20}P_2 = \frac{20!}{(20-2)!} = \frac{20!}{18!} = \frac{20 \cdot 19 \cdot \cancel{18!}}{\cancel{18!}} = 20 \cdot 19 = 380$$

UNDERSTAND A **combination** is an arrangement of items or events in which the order does not matter. The formula for finding the number of combinations of n objects taken r at a time is:

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

Suppose a dinner entrée is served with 2 sides from a menu of 5 choices. All of the food comes together, so order is not important. Fries and a salad is the same as a salad and fries. This combination may be expressed as "5 choose 2" and is represented by ${}_5C_2$

$${}_5C_2 = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2!(\cancel{3!})} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$$

Connect

Given the letters A, B, C, and D, determine the number of possible sequences of two letters if the order matters and the letters cannot be reused. Use a table and a formula and then compare the results.

1 Make a table and count the number of possibilities.

List the first choices in the top row and the second choices in the first column. Fill in the possible sequences. Remember that letters cannot be used twice.

	A	B	C	D
A		BA	CA	DA
B	AB		CB	DB
C	AC	BC		DC
D	AD	BD	CD	

Counting the boxes, there are 12 possible sequences.

2 Determine which formula to use.

It is given that the order of the letters matters, so AB is not the same sequence as BA. This means we are counting permutations.

The formula for the number of permutations of n objects taken r at a time is ${}_nP_r = \frac{n!}{(n-r)!}$.

3 Use the formula and compare the results.

There are 4 letters and we are choosing 2, so find ${}_4P_2$.

$${}_4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2!}{2!} = 4 \cdot 3 = 12$$

▶ Using both the table and the formula, we find that there are 12 possible sequences of two letters.

TRY

Suppose that the order of the letters does not matter. Look at the table again. How many possible combinations are there? Compare this to the result from the formula for ${}_nC_r$.

EXAMPLE A Ingrid will choose a group of 3 marbles at random from a bag. The bag contains one marble of each of the following colors: blue, green, yellow, red, orange, purple, black, and gray. How many different groups of 3 marbles can Ingrid choose?

1 Determine if the groups are permutations or combinations.

Imagine if Ingrid chooses a blue marble, then a red, then a purple. Compare this to choosing purple then blue then red. In the end, Ingrid has the same group of marbles in both scenarios: blue, red, and purple. Thus, the groups of marbles are combinations.

2 Set up a formula to find the number of combinations.

The formula for finding the number of combinations of n objects taken r at a time is ${}_nC_r = \frac{n!}{r!(n-r)!}$.

There are a total of 8 marbles, and Ingrid is choosing a group of 3. Therefore, we must find ${}_8C_3$.

3 Evaluate the formula.

$${}_8C_3 = \frac{8!}{3!(8-3)!}$$
$$= \frac{8!}{(3!)(5!)}$$
$$= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{(3!)(\cancel{5!})}$$
$$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$$
$$= 56$$

▶ There are 56 ways for Ingrid to choose a combination of 3 marbles from the bag.

DISCUSS

What is the probability that Ingrid will choose a group having a black marble, a green marble, and a gray marble?

Problem Solving

READ

The Pythagoras High School basketball team has twelve players. Six of those players are seniors. Two seniors will be chosen as team captains. How many different ways can the team choose two captains?

PLAN

The captains are equal in rank, so choosing Terry and Sam is the same as choosing Sam and Terry. The order in which the captains are chosen ____________ matter.

Therefore, we must use the formula for ___________________, which is

$_n\_\_\_\_\_\__r$ = ___________________.

How many students on the team are eligible to be a captain? __________

How many captains will be chosen? _________

So, find _____*C*_____.

SOLVE

Substitute ____ for *n* and _____ for *r* in the formula and evaluate.

____*C*_____ = _________________________________ = ____________

CHECK

Let the letters A, B, C, D, E, and F represent the six seniors on the team.

Would AB and BA represent different possible outcomes or the same outcome?

____

List every possible group of captains.

AB, AC, AD, AE, _______________________________________

How many groups are on the list? __________

▶ The team can choose from _____________ possible combinations of team captains.

Practice

Determine whether order matters for each situation. If order matters, write *permutation*. If order does not matter, write *combination*.

1. choosing 4 students from a class of 18 to pass out supplies ____________________

2. the numbers and letters on a license plate ____________________

3. buying ingredients for a salad ____________________

4. choosing the first, second, and third place winners of a contest ____________________

Evaluate each expression.

5. 4!

____

6. $\frac{7!}{3!}$

____

7. $\frac{10!}{8!}$

____

Choose the best answer.

8. There are 9 books on a shelf. If books are chosen at random, how many different groups of 3 books could be chosen?

A. 27

B. 84

C. 504

D. 60,480

9. To access a Web site, users must choose a 4-digit numerical PIN (Personal Identification Number) in which a digit may not be used more than once. How many unique PINs are possible?

A. 36

B. 126

C. 5,040

D. 15,120

Evaluate each combination or permutation.

10. ${}_9P_4$ = __________

11. ${}_{100}P_3$ = __________

12. ${}_7C_6$ = __________

13. ${}_{50}C_2$ = __________

Solve.

14. Debra must read 6 books over the summer for next year's English class. In how many different orders can she read the 6 books? Express your answer as a factorial and as a number.

____

15. For a jury trial, 5 people out of a panel of 11 people will be chosen as jurors. How many different groups of 5 jurors are possible?

____

16. Eighteen gymnasts are competing for the gold, silver, and bronze medals. How many ways can the medals be awarded among the gymnasts?

____

17. **DERIVE** Suppose that Alan, Brianna, and Cris are starting a band. One will play drums, one will play guitar, and one will sing. How many different configurations can the band have?

____

Write out a formula to solve this problem:

$${}_3P_3 = \frac{\square!}{(\square - \square)!} = \frac{\square!}{(\square)!}$$

Suppose a bag contains 5 marbles, one of each color: red, orange, yellow, green, and blue.

How many different groups of 5 marbles could you draw from the bag? __________

Write out a formula to solve this problem:

$${}_5C_5 = \frac{\square!}{\square!(\square - \square)!} = \frac{\square!}{\square!(\square)!} = \frac{\square}{\square!}$$

Based on these formulas, what does 0! equal? 0! = __________

18. **EXTEND** From her class of 12 students, Mrs. Albanez will select one player and one alternate at random for the math team. How many possibilities are there for Mrs. Albanez to select a player and an alternate? ____________________

What is the probability that she will choose the student with the highest grade as the player and the student with the second-highest grade as the alternate? Give your answer as a percent rounded to the nearest hundredth. ____________________