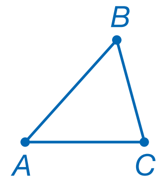
LESSON 7

Using Congruence to Prove Theorems

EXAMPLE A Prove that the sum of the measures of the interior angles of a triangle is 180°.

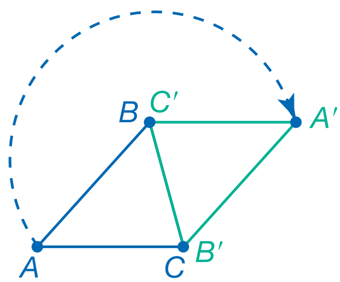
1 Draw a triangle.

Draw a triangle and label its vertices *A*, *B*, and *C*.



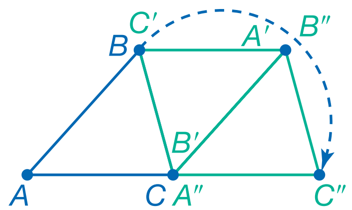
2 Produce an image and join two angles.

Rotate $\triangle ABC$ 180° to produce congruent image $\triangle A'B'C'$. If necessary, translate the image so that point B' lies on point C.



3 Produce another image and join all three angles.

Rotate $\triangle A'B'C'$ 180° to produce another congruent image, $\triangle A''B''C''$. This image can also be created by translating $\triangle ABC$ to the right. Translate the image so that point A'' lies on points C and B'.



When the three triangles are combined as shown, $\overline{AC}$ and $\overline{A''C''}$ are collinear, so

$m\angle A'' + m\angle B' + m\angle C = 180°$

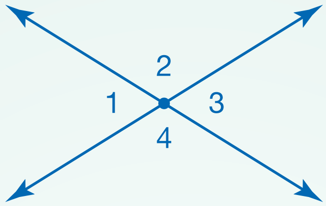
Since all three triangles are congruent, $m\angle B' = m\angle B$ and $m\angle A'' = m\angle A$.

▶ $m\angle A + m\angle B + m\angle C = 180°$

DISCUSS

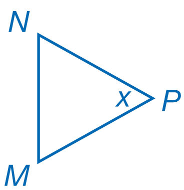
Would this proof also work for a right triangle? An obtuse triangle?

EXAMPLE B Non-adjacent angles formed by two intersecting lines are called vertical angles. Angles 1 and 3 are vertical angles. Use a 180° rotation of a triangle to prove that vertical angles are congruent.

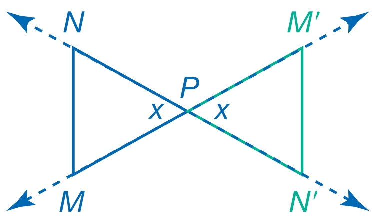


1 Draw a triangle.

Draw $\triangle MNP$. Let x represent the measure of $\angle P$.



2 Rotate $\triangle MNP$ 180° about point P.



Recall that after a rotation of 180°, corresponding line segments are either parallel or collinear. Corresponding segments $\overline{MP}$ and $\overline{M'P}$ both contain point P, so they must be collinear. Similarly, segments $\overline{NP}$ and $\overline{N'P}$ must be collinear. So, $\angle MPN$ and $\angle M'PN'$ are vertical angles.

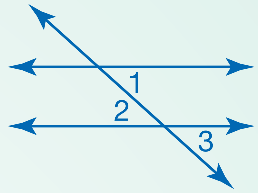
$\angle MPN$ and $\angle M'PN'$ are corresponding parts of congruent triangles MPN and $M'PN'$. Corresponding parts of congruent triangles are congruent (CPCTC).

▶ Vertical angles $\angle MPN$ and $\angle M'PN'$ are congruent.

TRY

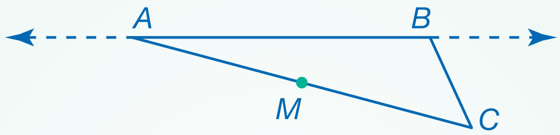
Are angles 2 and 4 also congruent? How could you prove this?

EXAMPLE C The diagram on the right shows two parallel lines cut by a transversal. Alternate interior angles are angles that lie between two parallel lines, are each adjacent to a different parallel line, are both adjacent to the transversal, and lie on opposite sides of the transversal. In the diagram on the right, $\angle 1$ and $\angle 2$ are alternate interior angles.

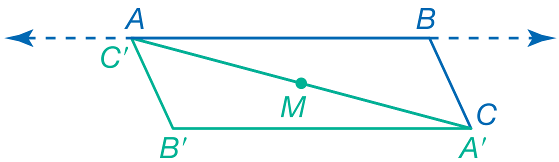


Corresponding angles are angles that lie on the same side of the transversal and the same side of different parallel lines. $\angle 1$ and $\angle 3$ are corresponding angles.

On $\triangle ABC$ below, point M is the midpoint of $\overline{AC}$. Use this triangle to prove that alternate interior angles are congruent and corresponding angles are congruent.



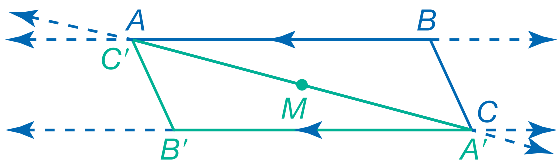
1 Rotate $\triangle ABC$ 180° around point M.



Rotating $\overline{AB}$ from the preimage produces corresponding side $\overline{A'B'}$ in the image. Corresponding sides are parallel after a rotation of 180°, so $\overline{AB} \parallel \overline{A'B'}$.

2 Extend $\overline{A'B'}$ and $\overline{AC}$ into lines.

Because $\overline{AB}$ is parallel to $\overline{A'B'}$, $\overleftrightarrow{AB}$ must be parallel $\overleftrightarrow{A'B'}$.



$\overleftrightarrow{AC}$ is a transversal that cuts across parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$.

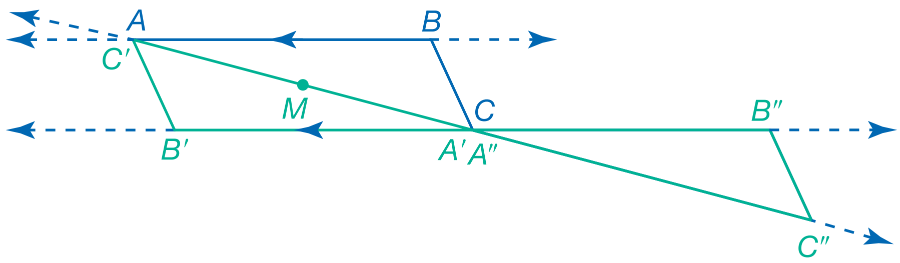
3 Compare alternate interior angles.

Angles BAC and $B'A'C'$ are alternate interior angles.

Rotation is a rigid motion, and rotating $\angle BAC$ produced $\angle B'A'C'$, so these two angles must be congruent.

▶ $\angle BAC \cong \angle B'A'C'$

4 Rotate $\angle A'B'C'$ 180° around point A'.



$\overline{A''B''}$ is collinear with $\overline{A'B'}$, so it is part of $\overleftrightarrow{A'B'}$. $\overline{A''C''}$ is collinear with $\overline{A'C'}$, so it is part of the transversal.

5 Compare alternate interior angles.

Angles BAC and $B''A''C''$ are corresponding angles.

Rotating $\angle B'A'C'$ produced $\angle B''A''C''$, so $\angle B'A'C'$ must be congruent to $\angle B''A''C''$.

Angle $B'A'C'$ is congruent to angle BAC, so angle $B''A''C''$ must also be congruent to angle BAC.

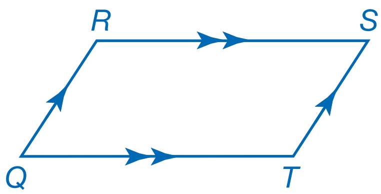
▶ $\angle B''A''C'' \cong \angle BAC$

CHECK

How could knowing about vertical angles help you determine that $\angle 1$ and $\angle 3$ are congruent?

Practice

Show that in parallelogram *QRST,* opposite angles ∠*Q* and ∠*S* are congruent by following the instructions and filling in the blanks.



1. Draw diagonal $\overline{RT}$ on the figure.

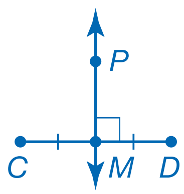
2. Rotating △*RQT* ________° around the midpoint of ________ will produce an image that covers △*TSR*.

3. ∠*Q* ≅ ∠*Q′* Corresponding angles are ________________________.

   ∠*Q′* ≅ ∠*S* CPCTC

   ∠*Q* ≅ ________ Transitive Property of Congruence

Line *PM* is the perpendicular bisector of $\overline{CD}$. Show that point *P* is equidistant from the endpoints of $\overline{CD}$ by following the instructions, filling in the blanks, and answering the questions below.



4. $\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{CD}$, so ∠*PMC* and ∠*PMD* are both ________________________ angles and $\overline{CM}$ is congruent to ________.

5. Draw line segments $\overline{CP}$ and $\overline{DP}$ on the figure. Triangles *CMP* and *DMP* share side ________.

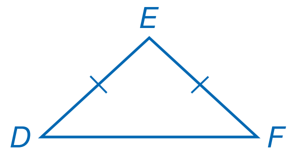
6. Which theorem or postulate can be used to prove that △*CMP* ≅ △*DMP*?

   ____

HINT: List the sides and angles of △*CMP* and △*DMP* that you know are congruent.

7. Corresponding sides of congruent triangles have equal lengths, so *PC* = ________. This means that point *P* is equidistant from points ________ and ________.

Use isosceles triangle *DEF* to prove that the base angles of an isosceles triangle are congruent. Follow the directions and fill in the blanks below.



8. On the figure above, draw the median from angle E to side $\overline{DF}$.

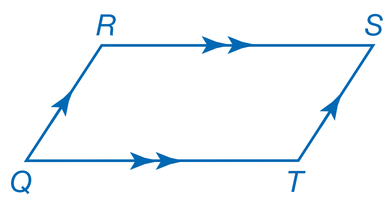
9. By definition, a median connects a vertex to the ______________ of the opposite side. So, point M divides $\overline{DF}$ into two congruent segments, $\overline{DM}$ and ______________. Mark these segments as congruent on the figure.

10. The marks on the diagram show that the triangles have two pairs of congruent ______________. Median $\overline{EM}$ is shared by both triangles, so by the reflexive property, $\overline{EM} \cong \overline{EM}$.

11. Three sides of $\triangle DME$ are congruent to three sides of ______________, so those triangles are congruent by ___________________________________.

12. Corresponding parts of congruent triangles are ____________________, so $\angle D \cong$ ________.

Use parallelogram *QRST* to prove that opposite sides of a parallelogram are congruent. Follow the directions, fill in the blanks, and answer the questions below.



13. Draw diagonal $\overline{QS}$ on the figure above to cut the parallelogram into two triangles.

14. Because *QRST* is a parallelogram, $\overleftrightarrow{RS}$ and ______________ are parallel lines. $\overleftrightarrow{QS}$ can be seen as a transversal. Angles *RSQ* and ______________ are alternate interior angles, so they must be congruent.

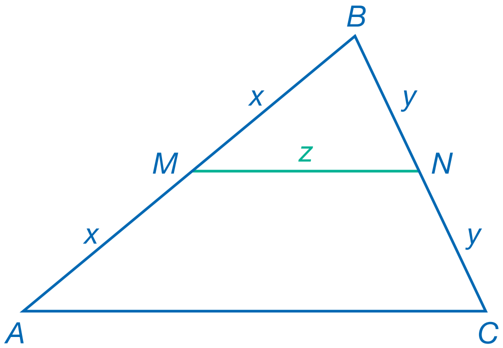
15. $\overleftrightarrow{QR}$ and ______________ are also parallel lines, and ______________ is also a transversal cutting across these lines. Angles *RQS* and *TSQ* are ____, so they must be congruent.

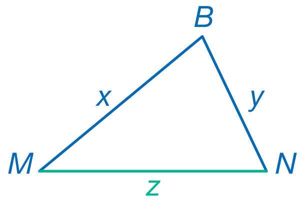
16. Triangles *RQS* and *TSQ* share side ______________, so they have at least one congruent side.

17. So, $\triangle RQS \cong \triangle TSQ$ by ______________ Theorem.

18. $\overline{RS}$ must be congruent to $\overline{TQ}$ and $\overline{RQ}$ must be congruent to $\overline{TS}$ because ______________________ parts of congruent triangles are ______________________.

Use $\triangle ABC$ and its top half, $\triangle MBN$, to show that the segment connecting the midpoints of two sides of a triangle is parallel to the third side and half the length of the third side. Follow the directions and answer the questions below.





19. Using $\triangle MBN$ on the right, draw the rotated image $\triangle M'B'N'$ after a 180° rotation of $\triangle MBN$ around the midpoint of $\overline{MN}$. Label the lengths of the sides using x, y, and z.

Then, draw the rotated image $\triangle M''B''N''$ after a 180° rotation of $\triangle M'B'N'$ around the midpoint of $\overline{B'N'}$. Label the lengths of the sides using x, y, and z.

Finally, draw the rotated image $\triangle M'''B'''N'''$ after a 180° rotation of $\triangle M'B'N'$ around the midpoint of $\overline{B'M'}$. Label the lengths of the sides using x, y, and z.

20. Find the lengths of the following segments in terms of x, y, and z:

$AB =$ ______ $M''B =$ ______ $CB =$ ______ $N'''B =$ ______

How does $\angle ABC$ compare to $\angle MBN$?

____

Use a postulate or theorem to prove that $\triangle ABC$ is congruent to $\triangle M''BN'''$, which you drew.

____

21. $\overline{M''N''}$ and $\overline{M'''N'''}$ can be combined to form $\overline{M''N'''}$. Compare $\overline{MN}$ to $\overline{M''N'''}$. Are the segments parallel, perpendicular, collinear, or none of these? Why?

____

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How does the length of $\overline{M''N'''}$ compare to the length of $\overline{MN}$?

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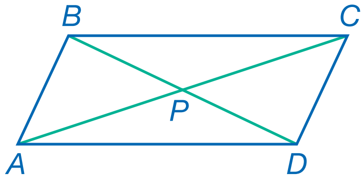
Complete the proofs.

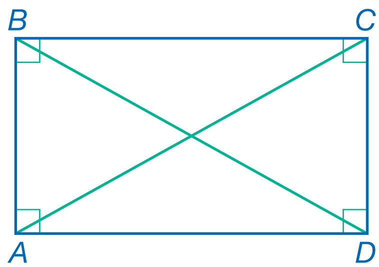
22. **PROVE** Parallelogram *ABCD* has diagonals $\overline{AC}$ and $\overline{BD}$. The diagonals intersect at point *P*.



Prove that the diagonals of a parallelogram bisect each other by filling the two-column proof below.

Statements	Reasons
1. $\overline{AC}$ and $\overline{BD}$ are diagonals of parallelogram *ABCD*.	Given
2. $\overline{BC} \parallel \overline{AD}$	Opposite sides of a ______________ are parallel.
3. $\angle BCA \cong \angle DAC$	______________________ angles are congruent.
4. $\angle CBD \cong \angle ADB$	______________________ angles are congruent.
5. $\overline{BC} \cong \overline{AD}$	Opposite sides of a __________________ are congruent.
6. $\triangle BPC \cong \triangle DPA$	____
7. $BP = DP$ and $AP = CP$	____

23. **PROVE** Rectangle *ABCD* has diagonals $\overline{AC}$ and $\overline{BD}$. Prove that the diagonals are congruent.



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