5-2 Perpendicular and Angle Bisectors

Mathematics Florida Standards

MAFS.912.G-CO.3.9 Prove theorems about lines and angles... points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints. **Also MAFS.912.G-SRT.2.5**

MP 1, MP 3, MP 4, MP 5, MP 8

Objective To use properties of perpendicular bisectors and angle bisectors

Confused? Try drawing a diagram to "straighten" yourself out.

MATHEMATICAL PRACTICES

SOLVE IT!

Getting Ready!

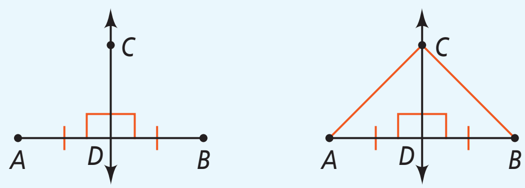
You hang a bulletin board over your desk using string. The bulletin board is crooked. When you straighten the bulletin board, what type of triangle does the string form with the top of the board? How do you know? Visualize the vertical line along the wall that passes through the nail. What relationships exist between this line and the top edge of the straightened bulletin board? Explain.

In the Solve It, you thought about the relationships that must exist in order for a bulletin board to hang straight. You will explore these relationships in this lesson.

Lesson Vocabulary
- equidistant
- distance from a point to a line

Essential Understanding There is a special relationship between the points on the perpendicular bisector of a segment and the endpoints of the segment.

In the diagram below on the left, $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$. $\overleftrightarrow{CD}$ is perpendicular to $\overline{AB}$ at its midpoint. In the diagram on the right, $\overline{CA}$ and $\overline{CB}$ are drawn to complete $\triangle CAD$ and $\triangle CBD$.



You should recognize from your work in Chapter 4 that $\triangle CAD \cong \triangle CBD$. So you can conclude that $\overline{CA} \cong \overline{CB}$, or that $CA = CB$. A point is **equidistant** from two objects if it is the same distance from the objects. So point C is equidistant from points A and B.

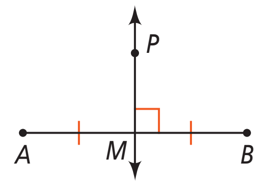
This suggests a proof of Theorem 5-2, the Perpendicular Bisector Theorem. Its converse is also true and is stated as Theorem 5-3.

take note

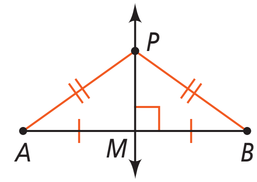
Theorem 5-2 Perpendicular Bisector Theorem

Theorem
If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

If . . .
$\overleftrightarrow{PM} \perp \overline{AB}$ and $MA = MB$



Then . . .
$PA = PB$

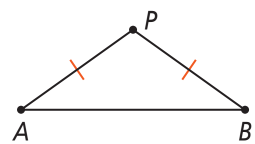


You will prove Theorem 5-2 in Exercise 32.

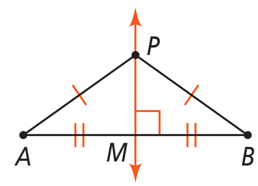
Theorem 5-3 Converse of the Perpendicular Bisector Theorem

Theorem
If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

If . . .
$PA = PB$



Then . . .
$\overleftrightarrow{PM} \perp \overline{AB}$ and $MA = MB$



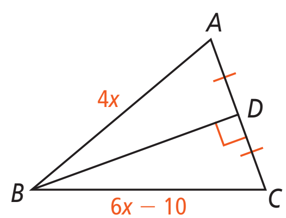
You will prove Theorem 5-3 in Exercise 33.

Problem 1 Using the Perpendicular Bisector Theorem

Think

How do you know $\overline{BD}$ is the perpendicular bisector of $\overline{AC}$?
The markings in the diagram show that $\overline{BD}$ is perpendicular to $\overline{AC}$ at the midpoint of $\overline{AC}$.

Algebra **What is the length of $\overline{AB}$?**



$\overline{BD}$ is the perpendicular bisector of $\overline{AC}$, so B is equidistant from A and C.

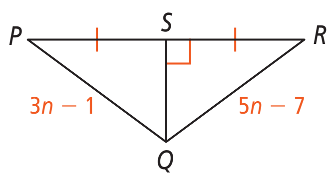
$BA = BC$	Perpendicular Bisector Theorem
$4x = 6x - 10$	Substitute $4x$ for BA and $6x - 10$ for BC.
$-2x = -10$	Subtract $6x$ from each side.
$x = 5$	Divide each side by -2.

Now find AB.

$AB = 4x$

$AB = 4(5) = 20$ Substitute 5 for x.

Got It? **1.** What is the length of $\overline{QR}$?

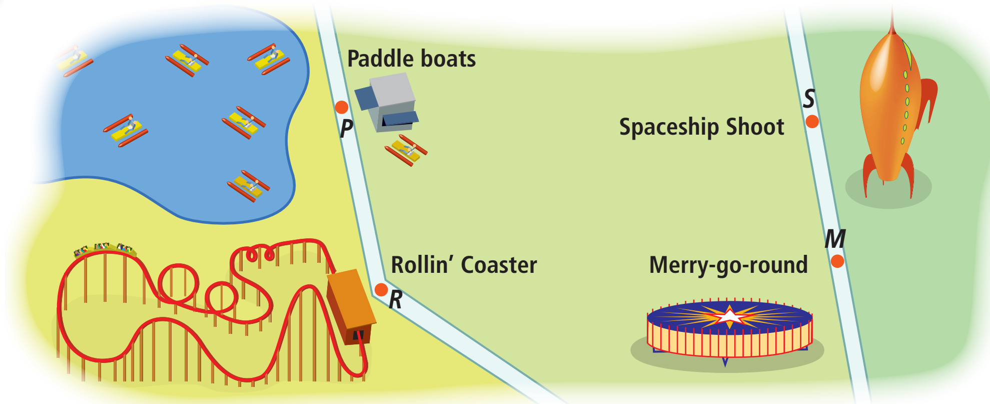


Plan

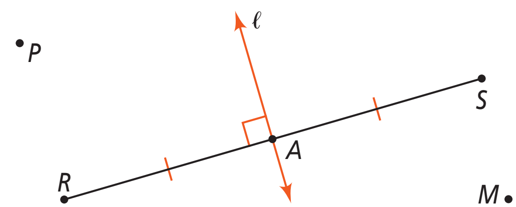
How do you find points that are equidistant from two given points?
By the Converse of the Perpendicular Bisector Theorem, points equidistant from two given points are on the perpendicular bisector of the segment that joins the two points.

Problem 2 Using a Perpendicular Bisector

A park director wants to build a T-shirt stand equidistant from the Rollin' Coaster and the Spaceship Shoot. What are the possible locations of the stand? Explain.



To be equidistant from the two rides, the stand should be on the perpendicular bisector of the segment connecting the rides. Find the midpoint A of $\overline{RS}$ and draw line ℓ through A perpendicular to $\overline{RS}$. The possible locations of the stand are all the points on line ℓ.

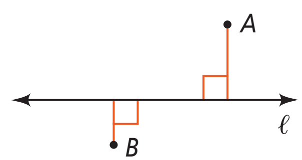


Got It? **2. a.** Suppose the director wants the T-shirt stand to be equidistant from the paddle boats and the Spaceship Shoot. What are the possible locations?

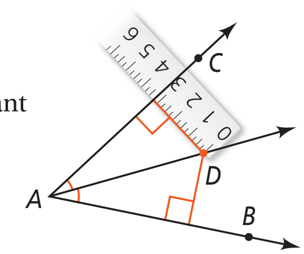
b. Reasoning Can you place the T-shirt stand so that it is equidistant from the paddle boats, the Spaceship Shoot, and the Rollin' Coaster? Explain.

Essential Understanding There is a special relationship between the points on the bisector of an angle and the sides of the angle.

The **distance from a point to a line** is the length of the perpendicular segment from the point to the line. This distance is also the length of the shortest segment from the point to the line. You will prove this in Lesson 5-6. In the figure at the right, the distances from A to ℓ and from B to ℓ are represented by the red segments.



In the diagram, $\overrightarrow{AD}$ is the bisector of $\angle CAB$. If you measure the lengths of the perpendicular segments from D to the two sides of the angle, you will find that the lengths are equal. Point D is equidistant from the sides of the angle.



take note

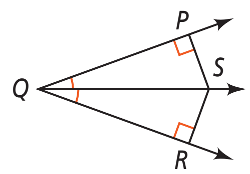
Theorem 5-4 Angle Bisector Theorem

Theorem

If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

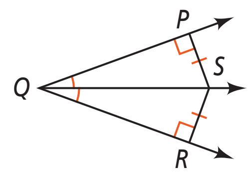
If . . .

$\overrightarrow{QS}$ bisects $\angle PQR$, $\overline{SP} \perp \overrightarrow{QP}$, and $\overline{SR} \perp \overrightarrow{QR}$



Then . . .

$SP = SR$



You will prove Theorem 5-4 in Exercise 34.

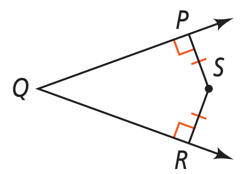
Theorem 5-5 Converse of the Angle Bisector Theorem

Theorem

If a point in the interior of an angle is equidistant from the sides of the angle, then the point is on the angle bisector.

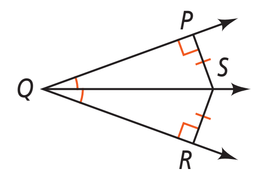
If . . .

$\overline{SP} \perp \overrightarrow{QP}$, $\overline{SR} \perp \overrightarrow{QR}$, and $SP = SR$



Then . . .

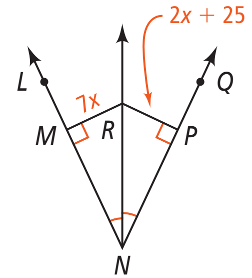
$\overrightarrow{QS}$ bisects $\angle PQR$



You will prove Theorem 5-5 in Exercise 35.

Problem 3 Using the Angle Bisector Theorem

Algebra **What is the length of $\overline{RM}$?**



Know

$\overrightarrow{NR}$ bisects $\angle LNQ$.

$\overline{RM} \perp \overrightarrow{NL}$ and $\overline{RP} \perp \overrightarrow{NQ}$.

Need

The length of $\overline{RM}$

Plan

Use the Angle Bisector Theorem to write an equation you can solve for x.

$RM = RP$	Angle Bisector Theorem
$7x = 2x + 25$	Substitute.
$5x = 25$	Subtract $2x$ from each side.
$x = 5$	Divide each side by 5.

Now find RM.

$RM = 7x$

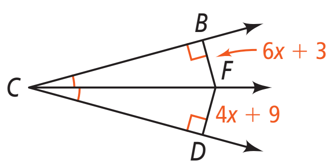
$= 7(5) = 35$ Substitute 5 for x.

Think

How can you use the expression given for RP to check your answer?

Substitute 5 for x in the expression $2x + 25$ and verify that the result is 35.

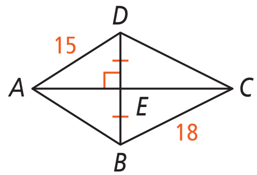
Got It? **3.** What is the length of $\overline{FB}$?



Lesson Check

Do you know HOW?

Use the figure at the right for Exercises 1–3.



1. What is the relationship between $\overline{AC}$ and $\overline{BD}$?

2. What is the length of $\overline{AB}$?

3. What is the length of $\overline{DC}$?

Do you UNDERSTAND? MATHEMATICAL PRACTICES

4. Vocabulary Draw a line and a point not on the line. Draw the segment that represents the distance from the point to the line.

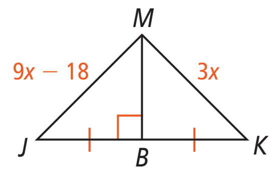
5. Writing Point P is in the interior of $\angle LOX$. Describe how you can determine whether P is on the bisector of $\angle LOX$ without drawing the angle bisector.

Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

A Practice

Use the figure at the right for Exercises 6–8.

See Problem 1.



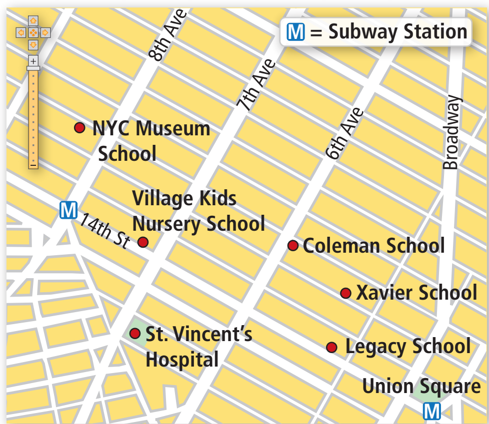
6. What is the relationship between $\overline{MB}$ and $\overline{JK}$?

7. What is value of x?

8. Find JM.

Reading Maps **For Exercises 9 and 10, use the map of a part of Manhattan.**

See Problem 2.



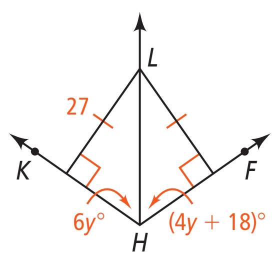
9. Which school is equidistant from the subway stations at Union Square and 14th Street? How do you know?

10. Is St. Vincent's Hospital equidistant from Village Kids Nursery School and Legacy School? How do you know?

11. Writing On a piece of paper, mark a point H for home and a point S for school. Describe how to find the set of points equidistant from H and S.

Use the figure at the right for Exercises 12–15.

See Problem 3.



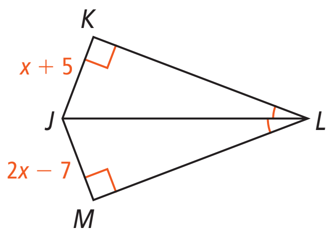
12. According to the diagram, how far is L from $\overrightarrow{HK}$? From $\overrightarrow{HF}$?

13. How is $\overrightarrow{HL}$ related to $\angle KHF$? Explain.

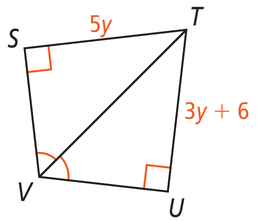
14. Find the value of y.

15. Find $m\angle KHL$ and $m\angle FHL$.

16. Algebra Find *x*, *JK*, and *JM*.

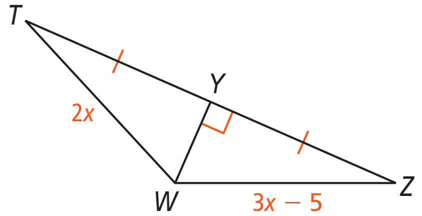


17. Algebra Find *y*, *ST*, and *TU*.



B Apply

Algebra **Use the figure at the right for Exercises 18–22.**



18. Find the value of *x*.

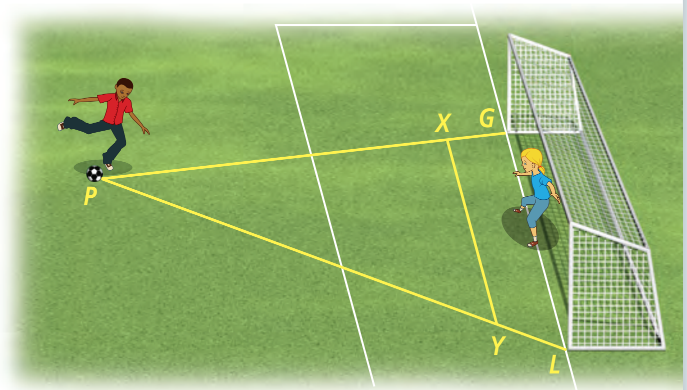
19. Find *TW*.

20. Find *WZ*.

21. What kind of triangle is $\triangle TWZ$? Explain.

22. If *R* is on the perpendicular bisector of $\overline{TZ}$, then *R* is __?__ from *T* and *Z*, or __?__ = __?__.

23. Think About a Plan In the diagram at the right, the soccer goalie will prepare for a shot from the player at point *P* by moving out to a point on $\overline{XY}$. To have the best chance of stopping the ball, should the goalie stand at the point on $\overline{XY}$ that lies on the perpendicular bisector of $\overline{GL}$ or at the point on $\overline{XY}$ that lies on the bisector of $\angle GPL$? Explain your reasoning.



- How can you draw a diagram to help?
- Would the goalie want to be the same distance from *G* and *L* or from $\overline{PG}$ and $\overline{PL}$?

24. a. Constructions Draw $\angle CDE$. Construct the angle bisector of the angle.

b. Reasoning Use the converse of the angle bisector theorem to justify your construction.

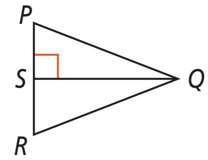
25. a. Constructions Draw $\overline{QR}$. Construct the perpendicular bisector of $\overline{QR}$ to construct $\triangle PQR$.

b. Reasoning Use the perpendicular bisector theorem to justify that your construction is an isosceles triangle.

26. Write Theorems 5-2 and 5-3 as a single biconditional statement.

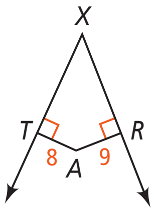
27. Write Theorems 5-4 and 5-5 as a single biconditional statement.

28. Error Analysis To prove that $\triangle PQR$ is isosceles, a student began by stating that since Q is on the segment perpendicular to $\overline{PR}$, Q is equidistant from the endpoints of $\overline{PR}$. What is the error in the student's reasoning?

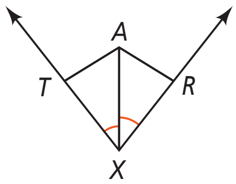


Writing **Determine whether A must be on the bisector of $\angle TXR$. Explain.**

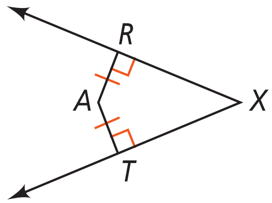
29.



30.



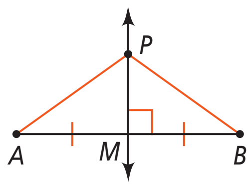
31.



32. Prove the Perpendicular Bisector Theorem.
Proof

Given: $\overleftrightarrow{PM} \perp \overline{AB}$, $\overleftrightarrow{PM}$ bisects $\overline{AB}$

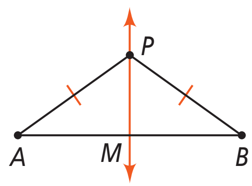
Prove: $AP = BP$



33. Prove the Converse of the Perpendicular Bisector Theorem.
Proof

Given: $PA = PB$ with $\overline{PM} \perp \overline{AB}$ at M.

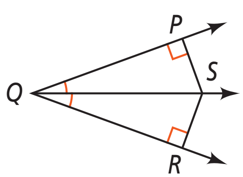
Prove: P is on the perpendicular bisector of $\overline{AB}$.



34. Prove the Angle Bisector Theorem.
Proof

Given: $\overrightarrow{QS}$ bisects $\angle PQR$, $\overline{SP} \perp \overrightarrow{QP}$, $\overline{SR} \perp \overrightarrow{QR}$

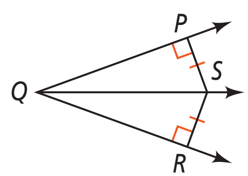
Prove: $SP = SR$



35. Prove the Converse of the Angle Bisector Theorem.
Proof

Given: $\overline{SP} \perp \overrightarrow{QP}$, $\overline{SR} \perp \overrightarrow{QR}$, $SP = SR$

Prove: $\overrightarrow{QS}$ bisects $\angle PQR$.



36. Coordinate Geometry Use points $A(6, 8)$, $O(0, 0)$, and $B(10, 0)$.

a. Write equations of lines ℓ and m such that $\ell \perp \overleftrightarrow{OA}$ at A and $m \perp \overleftrightarrow{OB}$ at B.

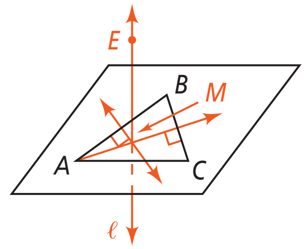
b. Find the intersection C of lines ℓ and m.

c. Show that $CA = CB$.

d. Explain why C is on the bisector of $\angle AOB$.

Challenge

37. *A, B,* and *C* are three noncollinear points. Describe and sketch a line in plane *ABC* such that points *A, B,* and *C* are equidistant from the line. Justify your response.



38. Reasoning *M* is the intersection of the perpendicular bisectors of two sides of $\triangle ABC$. Line ℓ is perpendicular to plane *ABC* at *M*. Explain why a point *E* on ℓ is equidistant from *A, B,* and *C.* (*Hint:* See page 48, Exercise 33. Explain why $\triangle EAM \cong \triangle EBM \cong \triangle ECM$.)

Standardized Test Prep

SAT/ACT

39. For $A(1, 3)$ and $B(1, 9)$, which point lies on the perpendicular bisector of $\overline{AB}$?

(A) (3, 3)　(B) (1, 5)　(C) (6, 6)　(D) (3, 12)

40. What is the converse of the following conditional statement?

If a triangle is isosceles, then it has two congruent angles.

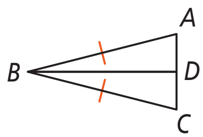
(F) If a triangle is isosceles, then it has two congruent sides.

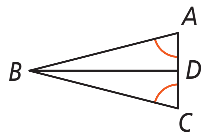
(G) If a triangle has congruent sides, then it is equilateral.

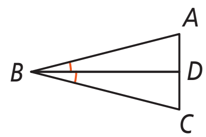
(H) If a triangle has two congruent angles, then it is isosceles.

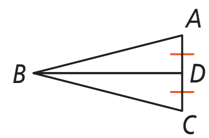
(I) If a triangle is not isosceles, then it does not have two congruent angles.

41. Which figure represents the statement $\overline{BD}$ bisects $\angle ABC$?

(A) 

(B) 

(C) 

(D) 

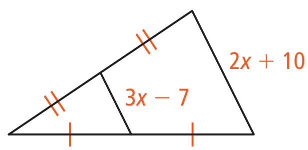
Short Response

42. The line $y = 7$ is the perpendicular bisector of the segment with endpoints $A(2, 10)$ and $B(2, k)$. What is the value of k? Explain your reasoning.

Mixed Review

43. Find the value of x in the figure at the right.

See Lesson 5-1.



44. $\angle 1$ and $\angle 2$ are complementary and $\angle 1$ and $\angle 3$ are supplementary. If $m\angle 2 = 30$, what is $m\angle 3$?

See Lesson 1-5.

Get Ready! **To prepare for Lesson 5-3, do Exercises 45–47.**

45. What is the slope of a line that is perpendicular to the line $y = -3x + 4$?

See Lesson 3-8.

46. Line ℓ is a horizontal line. What is the slope of a line perpendicular to ℓ?

47. Describe the line $x = 5$.