5-5 Indirect Proof

Mathematics Florida Standards

Extends MAFS.912.G-CO.3.10 Prove theorems about triangles . . .

MP 1, MP 3, MP 4

Objective To use indirect reasoning to write proofs

MATHEMATICAL PRACTICES

SOLVE IT! Getting Ready!

The goal of this game is to fill in the empty squares with numbers. The numbers 1, 2, 3, and 4 must appear once in each row and once in each column. Copy and complete the games on a piece of paper.

Game A

1			2
4		1	
	4		
		2	

Game B

			3
	1		
2		1	
	4		

Lesson Vocabulary
- indirect reasoning
- indirect proof

In the Solve It, you can conclude that a square must contain a certain number if you can eliminate the other three numbers as possibilities. This type of reasoning is called indirect reasoning. In **indirect reasoning,** all possibilities are considered and then all but one are proved false. The remaining possibility must be true.

Essential Understanding You can use indirect reasoning as another method of proof.

A proof involving indirect reasoning is an **indirect proof.** Often in an indirect proof, a statement and its negation are the only possibilities. When you see that one of these possibilities leads to a conclusion that contradicts a fact you know to be true, you can eliminate that possibility. For this reason, indirect proof is sometimes called *proof by contradiction.*

take note

Key Concept Writing an Indirect Proof

Step 1 State as a temporary assumption the opposite (negation) of what you want to prove.

Step 2 Show that this temporary assumption leads to a contradiction.

Step 3 Conclude that the temporary assumption must be false and that what you want to prove must be true.

In the first step of an indirect proof you assume as true the opposite of what you want to prove.

Problem 1 Writing the First Step of an Indirect Proof

Suppose you want to write an indirect proof of each statement. As the first step of the proof, what would you assume?

A An integer n is divisible by 5.

The opposite of "is divisible by" is "is not divisible by."
Assume temporarily that n is not divisible by 5.

Think

How do you find the opposite of a statement?
Write the negation of the statement. This often involves adding or removing the word *not*.

B You do not have soccer practice today.

The opposite of "do not have" is "do have."
Assume temporarily that you do have soccer practice today.

Got It? **1.** Suppose you want to write an indirect proof of each statement. As the first step of the proof, what would you assume?

a. $\triangle BOX$ is not acute.

b. At least one pair of shoes you bought cost more than $25.

To write an indirect proof, you have to be able to identify a contradiction.

Problem 2 Identifying Contradictions

Think

How do you know that two statements contradict each other?
A statement contradicts another statement if it is impossible for both to be true at the same time.

Which two statements contradict each other?

I. $\overline{FG} \parallel \overline{KL}$ **II.** $\overline{FG} \cong \overline{KL}$ **III.** $\overline{FG} \perp \overline{KL}$

Segments can be parallel and congruent. Statements I and II do not contradict each other.

Segments can be congruent and perpendicular. Statements II and III do not contradict each other.

Parallel segments do not intersect, so they cannot be perpendicular. Statements I and III contradict each other.

Got It? **2. a.** Which two statements contradict each other?

I. $\triangle XYZ$ is acute.
II. $\triangle XYZ$ is scalene.
III. $\triangle XYZ$ is equiangular.

b. Reasoning Statements I and II below contradict each other. Statement III is the negation of Statement I. Are Statements II and III equivalent? Explain your reasoning.

I. $\triangle ABC$ is scalene.
II. $\triangle ABC$ is equilateral.
III. $\triangle ABC$ is not scalene.

Proof

Problem 3 Writing an Indirect Proof

Given: $\triangle ABC$ is scalene.

Prove: $\angle A$, $\angle B$, and $\angle C$ all have different measures.

Think	Write
Assume temporarily the opposite of what you want to prove.	Assume temporarily that two angles of $\triangle ABC$ have the same measure. Assume that $m\angle A = m\angle B$.
Show that this assumption leads to a contradiction.	By the Converse of the Isosceles Triangle Theorem, the sides opposite $\angle A$ and $\angle B$ are congruent. This contradicts the given information that $\triangle ABC$ is scalene.
Conclude that the temporary assumption must be false and that what you want to prove must be true.	The assumption that two angles of $\triangle ABC$ have the same measure must be false. Therefore, $\angle A$, $\angle B$, and $\angle C$ all have different measures.

Got It? **3. Given:** $7(x + y) = 70$ and $x \neq 4$.
Prove: $y \neq 6$

Lesson Check

Do you know HOW?

1. Suppose you want to write an indirect proof of the following statement. As the first step of the proof, what would you assume?

Quadrilateral *ABCD* has four right angles.

2. Write a statement that contradicts the following statement. Draw a diagram to support your answer.

Lines *a* and *b* are parallel.

Do you UNDERSTAND? MATHEMATICAL PRACTICES

3. Error Analysis A classmate began an indirect proof as shown below. Explain and correct your classmate's error.

~~Given: $\triangle ABC$~~
~~Prove: $\angle A$ is obtuse.~~
~~Assume temporarily that $\angle A$ is acute.~~

Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

A Practice

Write the first step of an indirect proof of the given statement.

See Problem 1.

4. It is raining outside.
5. $\angle J$ is not a right angle.
6. $\triangle PEN$ is isosceles.
7. At least one angle is obtuse.
8. $\overline{XY} \cong \overline{AB}$
9. $m\angle 2 > 90$

Identify the two statements that contradict each other.

See Problem 2.

10. **I.** $\triangle PQR$ is equilateral.
II. $\triangle PQR$ is a right triangle.
III. $\triangle PQR$ is isosceles.

11. **I.** $\ell \parallel m$
II. ℓ and m do not intersect.
III. ℓ and m are skew.

12. **I.** Each of the two items that Val bought costs more than \$10.
II. Val spent \$34 for the two items.
III. Neither of the two items that Val bought costs more than \$15.

13. **I.** In right $\triangle ABC$, $m\angle A = 60$.
II. In right $\triangle ABC$, $\angle A \cong \angle C$.
III. In right $\triangle ABC$, $m\angle B = 90$.

14. Developing Proof Fill in the blanks to prove the following statement.

See Problem 3.

If the Yoga Club and Go Green Club together have fewer than 20 members and the Go Green Club has 10 members, then the Yoga Club has fewer than 10 members.

Given: The total membership of the Yoga Club and the Go Green Club is fewer than 20. The Go Green Club has 10 members.

Prove: The Yoga Club has fewer than 10 members.

Proof: Assume temporarily that the Yoga Club has 10 or more members.
This means that together the two clubs have **a.** __?__ members.
This contradicts the given information that **b.** __?__.
The temporary assumption is false. Therefore, it is true that **c.** __?__.

15. Developing Proof Fill in the blanks to prove the following statement.

In a given triangle, $\triangle LMN$, there is at most one right angle.

Given: $\triangle LMN$

Prove: $\triangle LMN$ has at most one right angle.

Proof: Assume temporarily that $\triangle LMN$ has more than one **a.** __?__. That is, assume that both $\angle M$ and $\angle N$ are **b.** __?__. If $\angle M$ and $\angle N$ are both right angles, then $m\angle M = m\angle N =$ **c.** __?__. By the Triangle Angle-Sum Theorem, $m\angle L + m\angle M + m\angle N =$ **d.** __?__. Use substitution to write the equation $m\angle L +$ **e.** __?__ $+$ **f.** __?__ $= 180$. When you solve for $m\angle L$, you find that $m\angle L =$ **g.** __?__. This means that there is no $\triangle LMN$, which contradicts the given statement. So the temporary assumption that $\triangle LMN$ has **h.** __?__ must be false. Therefore, $\triangle LMN$ has **i.** __?__.

B Apply

16. History Use indirect reasoning to eliminate all but one of the following answers. In what year was George Washington born?

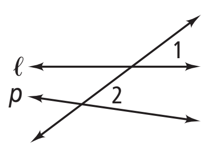
(A) 1492 (B) 1732 (C) 1902 (D) 2002

17. Think About a Plan Write an indirect proof.

Proof

Given: $\angle 1 \not\cong \angle 2$

Prove: $\ell \nparallel p$



- What assumption should be the first step of your proof?
- In the figure, what type of angle pair do $\angle 1$ and $\angle 2$ form?

Write the first step of an indirect proof of the given statement.

18. If a number n ends in 5, then it is not divisible by 2.

19. If point X is on the perpendicular bisector of $\overline{AB}$, then $\overline{XB} \cong \overline{XA}$.

20. If a transversal intersects two parallel lines, then alternate exterior angles are congruent.

21. Reasoning Identify the two statements that contradict each other.

I. The orthocenter of $\triangle JRK$ is on the triangle.

II. The centroid of $\triangle JRK$ is inside the triangle.

III. $\triangle JRK$ is an obtuse triangle.

Write an indirect proof.

22. Proof **Given:** $\triangle ABC$ with $BC > AC$
Prove: $\angle A \ncong \angle B$

23. Proof **Given:** $\triangle XYZ$ is isosceles.
Prove: Neither base angle is a right angle.

Writing **For Exercises 24 and 25, write a convincing argument that uses indirect reasoning.**

STEM **24. Chemistry** Ice is forming on the sidewalk in front of Toni's house. Show that the temperature of the sidewalk surface must be 32°F or lower.

25. Show that an obtuse triangle cannot contain a right angle.

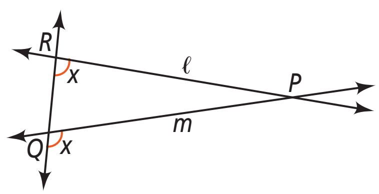
26. Error Analysis Your friend wants to prove indirectly that $\triangle ABC$ is equilateral. For a first step, he writes, "Assume temporarily that $\triangle ABC$ is scalene." What is wrong with your friend's statement? How can he correct himself?

27. Literature In Arthur Conan Doyle's story "The Sign of the Four," Sherlock Holmes talks to his friend Watson about how a culprit enters a room that has only four entrances: a door, a window, a chimney, and a hole in the roof.

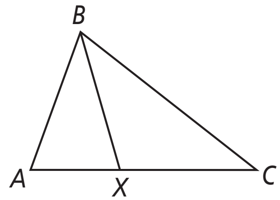
"You will not apply my precept," he said, shaking his head. "How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth? We know that he did not come through the door, the window, or the chimney. We also know that he could not have been concealed in the room, as there is no concealment possible. Whence, then, did he come?"

How did the culprit enter the room? Explain.

28. Proof In the figure at the right, $\overleftrightarrow{RQ}$ intersects lines ℓ and m to form congruent corresponding angles, and lines ℓ and m intersect at point P. Use the figure and the Triangle Angle-Sum Theorem to write an indirect proof of Theorem 3-4, the Converse of the Corresponding Angles Theorem.



Challenge Use the figure at the right for Exercises 29 and 30.



29. Given: $\triangle ABC$ is scalene, $m\angle ABX = 36$, $m\angle CBX = 36$
Proof **Prove:** $\overline{XB}$ is not perpendicular to $\overline{AC}$.

30. Given: $\triangle ABC$ is scalene, $m\angle ABX = 36$, $m\angle CBX = 36$
Proof **Prove:** $\overline{AX} \ncong \overline{XC}$

Standardized Test Prep

SAT/ACT

31. What temporary assumption is the first step of the following indirect proof?

Given: The sides of $\triangle SFK$ measure 3 cm, 4 cm, and 5 cm.

Prove: The orthocenter of $\triangle SFK$ is on the triangle.

(A) The incenter of $\triangle SFK$ is inside the triangle.

(B) The orthocenter of $\triangle SFK$ is inside the triangle.

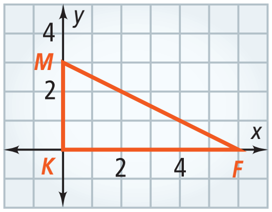
(C) The centroid of $\triangle SFK$ is outside the triangle.

(D) The orthocenter of $\triangle SFK$ is inside or outside the triangle.

32. $\triangle LMN \cong \triangle OPQ$, $m\angle L = 39$, and $m\angle P = 61$. What is $m\angle Q$?

(F) 39 (G) 80 (H) 100 (I) 141

33. In the diagram, what are the coordinates of the circumcenter of $\triangle KMF$?



(A) (1.5, 1.5) (C) (3, 1.5)

(B) (6, 0) (D) (0, 0)

Short Response

34. For what types of triangles are the centroid, circumcenter, incenter, and orthocenter all inside the triangle? Explain.

Mixed Review

35. The distances from the centroid of a triangle to its vertices are 16 cm, 17 cm, and 18 cm. What is the length of the shortest median? See Lesson 5-4.

36. The orthocenter of isosceles $\triangle ABC$ lies outside the triangle and $m\angle A = 30$. What are the measures of the two other angles?

37. You think, "If I leave home at 7:10, I'll catch the 7:25 bus. If I catch the 7:25 bus, I'll get to school before class starts. I am leaving and it's 7:10, so I'll get to school on time." Which law of deductive reasoning are you using? See Lesson 2-4.

Get Ready! **To prepare for Lesson 5-6, do Exercises 38–40.**

Graph $\triangle ABC$. List the sides in order from shortest to longest. See Lesson 1-7.

38. $A(5, 0)$, $B(0, 8)$, $C(0, 0)$ **39.** $A(2, 4)$, $B(-5, 1)$, $C(0, 0)$ **40.** $A(3, 0)$, $B(4, 3)$, $C(8, 0)$